MATH 221, Lecture 9, September 27, 2000

Leftovers:

$\log_a x$ is the inverse function to a^x.

So $\log_a a^{\sqrt{7}\pi i \sin 32} = \sqrt{7}\pi i \sin 32$

and $a^{\log_a(\sqrt{7}\pi i \sin 32)} = \sqrt{7}\pi i \sin 32$

Example Find $\frac{dy}{dx}$ when $y = \log_x 10$.

So $x^y = x^{\log_x 10} = 10$.

Take the derivative:

$$\frac{d x^y}{dx} = \frac{d(e^{\ln x})^y}{dx} = \frac{d e^{y\ln x}}{dx} = e^{y \ln x}\left(y\frac{1}{x} + \frac{dy}{dx}\ln x\right).$$

So $e^{y\ln x}\left(\frac{y}{x} + \frac{dy}{dx}\ln x\right) = 0$ (since $x^y = 10$ and $\frac{d\,10}{dx} = 0$.)

Solve for $\frac{dy}{dx}$.

$$e^{y\ln x}\frac{dy}{dx}\ln x = \frac{-e^{y\ln x}y}{x}$$

$$\frac{dy}{dx} = \frac{-e^{y\ln x}y}{x e^{y\ln x}\ln x} = \frac{-y}{x\ln x} = \frac{-\log_x 10}{x \ln x}.$$

Example Find the third derivative of 2^x with respect to x.

$y = 2^x$

$$\frac{dy}{dx} = \frac{d\,2^x}{dx} = \frac{d(e^{\ln 2})^x}{dx} = \frac{d\,e^{x\ln 2}}{dx}$$

$$= e^{x\ln 2}(\ln 2) = (e^{\ln 2})^x \ln 2 = 2^x \ln 2.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,2^x\ln 2}{dx} = \ln 2 \cdot 2^x \ln 2 = (\ln 2)^2 2^x.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left((\ln 2)^2 2^x\right) = (\ln 2)^2 2^x \ln 2$$

$$= (\ln 2)^3 2^x.$$

Example If $y = a\cos(\ln x) + b\sin(\ln x)$ show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$$\frac{dy}{dx} = a(-\sin(\ln x))\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$= -a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1}.$$

$$\frac{d^2y}{dx^2} = -a\cos(\ln x)\frac{1}{x}x^{-1} + -a\sin(\ln x)(-1)x^{-2}$$

$$+ -b\sin(\ln x)\frac{1}{x}x^{-1} + b\cos(\ln x)(-1)x^{-2}$$

$$= \frac{-a\cos(\ln x) + a\sin(\ln x) - b\sin(\ln x) - b\cos(\ln x)}{x^2}$$

$$= \frac{1}{x^2}\left((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\right)$$

So

$$LHS = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y$$

$$= x^2\frac{1}{x}\left((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\right)$$

$$+ x\left(-a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1}\right)$$

$$+ a\cos(\ln x) + b\sin(\ln x)$$

$$= (a-b)\sin(\ln x) - (a+b)\cos(\ln x)$$
$$- a\sin\ln x \quad + b\cos\ln(x)$$
$$+ b\sin\ln x \quad + a\cos\ln x$$

$$= 0.$$

<u>Example</u> Find $\frac{dy}{dx}$ when $a\sin(xy) + b\cos\left(\frac{x}{y}\right) = 0$.

Take the derivative:

$$0 = a\cos(xy)\left(x\frac{dy}{dx} + 1\cdot y\right) + -b\sin\left(\frac{x}{y}\right)\left(x(-1)y^{-2}\frac{dy}{dx} + 1\cdot y^{-1}\right)$$

$$= a\cos(xy)\,x\frac{dy}{dx} + a\cos(xy)\,y$$
$$+ b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} - b\sin\left(\frac{x}{y}\right)y^{-1}.$$

Solve for $\frac{dy}{dx}$.

$$a\cos(xy)\,x\frac{dy}{dx} + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} = a\cos(xy)\,y - b\sin\left(\frac{x}{y}\right)y^{-1}$$

$$\frac{dy}{dx} = \frac{a\cos(xy)\,y - b\sin\left(\frac{x}{y}\right)y^{-1}}{a\cos(xy)\,x + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}}$$

$$= \frac{a\cos(xy)\,y^2 - b\sin\left(\frac{x}{y}\right)}{a\cos(xy)\,xy + b\sin\left(\frac{x}{y}\right)\frac{x}{y}}$$

$$= \frac{a\cos(xy)\,y^3 - b\sin\left(\frac{x}{y}\right)}{a\cos(xy)\,xy^2 + b\sin\left(\frac{x}{y}\right)x}.$$

<u>Example</u> Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{a}{x}\right)\cdot\cot^{-1}\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = \tan^{-1}\left(\frac{a}{x}\right)\left(\frac{-1}{1+\left(\frac{x}{a}\right)^2}\right)\frac{1}{a} + \frac{1}{1+\left(\frac{a}{x}\right)^2}(-1)ax^{-2}\cot^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)}{a + \frac{x^2}{a}} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2 + a^2}$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)a}{a^2 + x^2} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2 + a^2}$$

$$= \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \cot^{-1}\left(\frac{x}{a}\right)\right)$$

If $\frac{a}{x} = \tan z$ then $\frac{x}{a} = \cot z$

and $z = \tan^{-1}\left(\frac{a}{x}\right) = \cot^{-1}\left(\frac{x}{a}\right)$.

So
$$\frac{dy}{dx} = \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{a}{x}\right)\right)$$
$$= \frac{-2a\tan^{-1}\left(\frac{a}{x}\right)}{a^2+x^2}.$$

<u>Example</u> Suppose

$$f = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Find the c's.

$$f(a) = c_0 + 0 + 0 + 0 + \cdots = c_0$$

$$\left.\frac{df}{dx}\right|_{x=a} = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \Big|_{x=a}$$
$$= c_1 + 0 + 0 + \cdots = c_1$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=a} = 0 + 2c_2 + 6c_3(x-a) + 4\cdot 3(x-a)^2 + \cdots \Big|_{x=a}$$
$$= 2c_2 + 0 + 0 + \cdots = 2c_2$$

So $c_2 = \frac{1}{2}\left(\left.\frac{d^2f}{dx^2}\right|_{x=a}\right)$

<u>Example</u> Find the function f such that

$$f(10) = 2, \quad \left.\frac{df}{dx}\right|_{x=10} = -3, \quad \left.\frac{d^2f}{dx^2}\right|_{x=10} = 2 \quad \left.\frac{d^3f}{dx^3}\right|_{x=10} = 1$$

and $\left.\frac{d^nf}{dx^n}\right|_{x=10} = 0$ for all $n > 3$.

Say $f = c_0 + c_1(x-10) + c_2(x-10)^2 + c_3(x-10)^3 + \cdots$

Then $f(10) = c_0 = 2$. So $c_0 = 2$.

$$-3 = \left.\frac{df}{dx}\right|_{x=10} = 0 + c_1 + 2c_2(x-10) + 3c_3(x-10)^2 + \cdots \Big|_{x=10}$$
$$= c_1 + 0 + 0 + 0 + \cdots = c_1.$$

So $c_1 = -3$.

$$2 = \left.\frac{d^2f}{dx^2}\right|_{x=10} = 2c_2 + 3\cdot 2c_3(x-10) + 4\cdot 3(x-10)^2 + \cdots \Big|_{x=10}$$
$$= 2c_2 + 0 + 0 + 0 + \cdots = 2c_2$$

So $c_2 = 1$

$$1 = \left.\frac{d^3f}{dx^3}\right|_{x=10} = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-10) + 5\cdot 4\cdot 3(x-10)^2c_5 + \cdots \Big|_{x=10}$$
$$= 6c_3 + 0 + 0 + 0 + \cdots$$

So $c_3 = \frac{1}{6}$.

$$0 = \left.\frac{d^4 f}{dx^4}\right|_{x=10} = 4\cdot3\cdot2c_4 + 5\cdot4\cdot3\cdot2c_5(x-10) + \cdots \Big|_{x=10}$$

$$= 4\cdot3\cdot2c_4 + 0 + 0 + \cdots = 4\cdot3\cdot2c_4$$

So $c_4 = 0$.

$$0 = \left.\frac{d^5 f}{dx^5}\right|_{x=10} = 5\cdot4\cdot3\cdot2c_5 + 6\cdot5\cdot4\cdot3\cdot2(x-10)c_6 + \cdots \Big|_{x=10}$$

$$= 5\cdot4\cdot3\cdot2c_5 + 0 + 0 + \cdots$$

$$= 5!\,c_5$$

So $c_5 = 0$.

Similarly $c_6 = 0$, $c_7 = 0$, $c_8 = 0, \ldots$

Since $\left.\frac{d^n f}{dx^n}\right|_{x=10} = 0$ for $n = 6, 7, 8, \ldots$

So $f = c_0 + c_1(x-10) + c_2(x-10)^2 + c_3(x-10)^3 + \cdots$

$$= 2 + -3(x-10) + 1\cdot(x-10)^2 + \tfrac{1}{6}(x-10)^3 + 0 + 0 + \cdots$$

$$= 2 - 3x + 30 + x^2 - 20x + 100 + \tfrac{1}{6}(x^3 - 30x^2 + 300x - 1000)$$

$$= \tfrac{1}{6}x^3 - 4x^2 + 27x + 132 - \tfrac{1000}{6}$$

So $f = \tfrac{1}{6}x^3 - 4x^2 + 27x - 61\tfrac{1}{3}$.