Inverse functions and their derivatives

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$\sqrt{x}$ is the function that undoes x^2. This means that

$$\sqrt{x^2} = x \qquad \text{and} \qquad (\sqrt{x})^2 = x.$$

$\ln x$ is the function that undoes e^x. This means that

$$\ln(e^x) = x \qquad \text{and} \qquad e^{\ln x} = x.$$

$\sin^{-1} x$ is the function that undoes $\sin x$. This means that

$$\sin^{-1}(\sin x) = x \qquad \text{and} \qquad \sin(\sin^{-1} x) = x.$$

$\cos^{-1} x$ is the function that undoes $\cos x$. This means that

$$\cos^{-1}(\cos x) = x \qquad \text{and} \qquad \cos(\cos^{-1} x) = x.$$

$\tan^{-1} x$ is the function that undoes $\tan x$. This means that

$$\tan^{-1}(\tan x) = x \qquad \text{and} \qquad \tan(\tan^{-1} x) = x.$$

$\cot^{-1} x$ is the function that undoes $\cot x$. This means that

$$\cot^{-1}(\cot x) = x \qquad \text{and} \qquad \cot(\cot^{-1} x) = x.$$

$\sec^{-1} x$ is the function that undoes $\sec x$. This means that

$$\sec^{-1}(\sec x) = x \qquad \text{and} \qquad \sec(\sec^{-1} x) = x.$$

$\csc^{-1} x$ is the function that undoes $\csc x$. This means that

$$\csc^{-1}(\csc x) = x \qquad \text{and} \qquad \csc(\csc^{-1} x) = x.$$

$\log_a x$ is the function that undoes a^x. This means that

$$\log_a(a^{\sqrt{7}\pi i \sin 32}) = \sqrt{7}\pi i \sin 32 \qquad \text{and} \qquad a^{\log_a(\sqrt{7}\pi i \sin 32)} = \sqrt{7}\pi i \sin 32.$$

WARNING: $\sin^{-1} x$ is VERY DIFFERENT from $(\sin x)^{-1}$. For example,

$$\sin^{-1} 0 = \sin^{-1}(\sin 0) = 0, \qquad \text{BUT} \qquad (\sin 0)^{-1} = \frac{1}{\sin 0} = \frac{1}{0} = \text{UNDEFINED}.$$

Example: Explain why $\ln 1 = 0$.

$$\ln 1 = \ln(e^0) = 0.$$

Example: Explain why $\ln(ab) = \ln a + \ln b$.

$$\ln(ab) = \ln(e^{\ln a} \cdot e^{\ln b}) = \ln(e^{\ln a + \ln b}) = \ln a + \ln b.$$

Example: Explain why $\ln\left(\frac{1}{a}\right) = -\ln a$.

$$\ln\left(\frac{1}{a}\right) = \ln\left(\frac{1}{e^{\ln a}}\right) = \ln\left(e^{-\ln a}\right) = -\ln a.$$

Example: Explain why $\ln\left(a^b\right) = b \ln a$.

$$\ln(a^b) = \ln\left(\left(e^{\ln a}\right)^b\right) = \ln\left(e^{b \ln a}\right) = b \ln a.$$

Thus

$$\begin{array}{lll} e^0 = 1 & \text{turns into} & \ln 1 = 0, \\ e^x e^y = e^{x+y} & \text{turns into} & \ln(ab) = \ln a + \ln b, \\ e^{-x} = \dfrac{1}{e^x} & \text{turns into} & \ln\left(\dfrac{1}{a}\right) = -\ln a, \quad \text{and} \\ (e^x)^y = e^{yx} & \text{turns into} & \ln(a^b) = b \ln a. \end{array}$$

Example: Explain why $\frac{d \ln x}{dx} = \frac{1}{x}$.

Since $\quad e^{\ln x} = x, \qquad \frac{d e^{\ln x}}{dx} = \frac{dx}{dx}.$

So $\quad e^{\ln x} \frac{d \ln x}{dx} = 1. \qquad$ So $\quad x \frac{d \ln x}{dx} = 1. \qquad$ So $\quad \frac{d \ln x}{dx} = \frac{1}{x}.$

Example: Find $\frac{d \sin^{-1} x}{dx}$.

Since $\quad \sin(\sin^{-1} x) = x, \qquad \frac{d \sin(\sin^{-1} x)}{dx} = \frac{dx}{dx}.$

So $\cos(\sin^{-1} x)\,\dfrac{d\sin^{-1} x}{dx} = 1.$ So $\dfrac{d\sin^{-1} x}{dx} = \dfrac{1}{\cos(\sin^{-1} x)}.$

So we would like to "simplify" $\cos(\sin^{-1} x)$.

Since $1 - \cos^2(\sin^{-1} x) = \sin^2(\sin^{-1} x)$, $1 - \big(\cos(\sin^{-1} x)\big)^2 = \big(\sin(\sin^{-1} x)\big)^2.$

So $1 - \big(\cos(\sin^{-1} x)\big)^2 = x^2.$ So $1 - x^2 = \big(\cos(\sin^{-1} x)\big)^2.$

So $\cos(\sin^{-1} x) = \sqrt{1-x^2}.$ So $\dfrac{d\sin^{-1} x}{dx} = \dfrac{1}{\cos(\sin^{-1} x)} = \dfrac{1}{\sqrt{1-x^2}}.$

Example: Find $\dfrac{d\cos^{-1} x}{dx}$.

Since $\cos(\cos^{-1} x) = x$, $\dfrac{d\cos(\cos^{-1} x)}{dx} = \dfrac{dx}{dx}.$

So $-\sin(\cos^{-1} x)\,\dfrac{d\cos^{-1} x}{dx} = 1.$ So $\dfrac{d\cos^{-1} x}{dx} = \dfrac{-1}{\sin(\cos^{-1} x)}.$

So we would like to "simplify" $\sin(\cos^{-1} x)$.

Since $1 - \sin^2(\cos^{-1} x) = \cos^2(\cos^{-1} x)$, $1 - \big(\sin(\cos^{-1} x)\big)^2 = \big(\cos(\cos^{-1} x)\big)^2.$

So $1 - \big(\sin(\cos^{-1} x)\big)^2 = x^2.$ So $1 - x^2 = \big(\sin(\cos^{-1} x)\big)^2.$

So $\sin(\cos^{-1} x) = \sqrt{1-x^2}.$ So $\dfrac{d\cos^{-1} x}{dx} = \dfrac{-1}{\sin(\cos^{-1} x)} = \dfrac{-1}{\sqrt{1-x^2}}.$

Example: Find $\dfrac{d\tan^{-1} x}{dx}$.

Since $\tan(\tan^{-1} x) = x$, $\dfrac{d\tan(\tan^{-1} x)}{dx} = \dfrac{dx}{dx}.$

So $\sec^2(\tan^{-1} x)\,\dfrac{d\tan^{-1} x}{dx} = 1.$ So $\dfrac{d\tan^{-1} x}{dx} = \dfrac{1}{\sec^2(\tan^{-1} x)}.$

So we would like to "simplify" $\sec^2(\tan^{-1} x)$.

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

So $\tan^2 x + 1 = \sec^2 x.$

So $\sec^2(\tan^{-1} x) = \tan^2(\tan^{-1} x) + 1 = \big(\tan(\tan^{-1} x)\big)^2 + 1 = x^2 + 1.$

So $\dfrac{d\tan^{-1} x}{dx} = \dfrac{1}{x^2+1}.$

Example: Find $\dfrac{d\cot^{-1} x}{dx}$.

Since $\cot(\cot^{-1} x) = x$, $\dfrac{d\cot(\cot^{-1} x)}{dx} = \dfrac{dx}{dx}.$

So $-\csc^2(\cot^{-1} x)\,\dfrac{d\cot^{-1} x}{dx} = 1.$ So $\dfrac{d\cot^{-1} x}{dx} = \dfrac{-1}{\csc^2(\cot^{-1} x)}.$

So we would like to "simplify" $\csc^2(\cot^{-1} x)$.

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

So $1 + \cot^2 x = \csc^2 x$.

So $\csc^2(\cot^{-1} x) = 1 + \cot^2(\cot^{-1} x) = 1 + \big(\cot(\cot^{-1} x)\big)^2 = 1 + x^2$.

So $\dfrac{d \cot^{-1} x}{dx} = \dfrac{-1}{1+x^2}$.

Example: Find $\dfrac{d \sec^{-1} x}{dx}$.

Since $\sec(\sec^{-1} x) = x$, $\dfrac{d \sec(\sec^{-1} x)}{dx} = \dfrac{dx}{dx}$.

So $\tan(\sec^{-1} x)\sec(\sec^{-1} x)\dfrac{d \sec^{-1} x}{dx} = 1$. So $\tan(\sec^{-1} x) \cdot x \cdot \dfrac{d \sec^{-1} x}{dx} = 1$.

So $\dfrac{d \sec^{-1} x}{dx} = \dfrac{1}{x \tan(\sec^{-1} x)}$.

So we would like to "simplify" $\tan(\sec^{-1} x)$.

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

So $\tan^2 x + 1 = \sec^2 x$.

So $\tan^2(\sec^{-1} x) + 1 = \sec^2(\sec^{-1} x)$. So $\big(\tan(\sec^{-1} x)\big)^2 + 1 = \big(\sec(\sec^{-1} x)\big)^2$.

So $\big(\tan(\sec^{-1} x)\big)^2 + 1 = x^2$. So $\tan(\sec^{-1} x) = \sqrt{x^2 - 1}$.

So $\dfrac{d \sec^{-1} x}{dx} = \dfrac{1}{x\sqrt{x^2 - 1}}$.

Example: Find $\dfrac{d \csc^{-1} x}{dx}$.

Since $\csc(\csc^{-1} x) = x$, $\dfrac{d \csc(\csc^{-1} x)}{dx} = \dfrac{dx}{dx}$.

So $-\csc(\csc^{-1} x)\cot(\csc^{-1} x)\dfrac{d \csc^{-1} x}{dx} = 1$. So $-x\cot(\csc^{-1} x)\dfrac{d \csc^{-1} x}{dx} = 1$.

So $\dfrac{d \csc^{-1} x}{dx} = \dfrac{-1}{x \cot(\csc^{-1} x)}$.

So we would like to "simplify" $\cot(\csc^{-1} x)$.

Since $\sin^2 x + \cos^2 x = 1$,

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

So $1 + \cot^2 x = \csc^2 x$.

So $1 + \cot^2(\csc^{-1} x) = \csc^2(\csc^{-1} x)$. So $1 + \big(\cot(\csc^{-1} x)\big)^2 = \big(\csc(\csc^{-1} x)\big)^2$.

So $1+\big(\cot(\csc^{-1}x)\big)^2 = x^2$. So $\cot(\csc^{-1}x) = \sqrt{x^2-1}$.

So $\dfrac{d\csc^{-1}x}{dx} = \dfrac{-1}{x\sqrt{x^2-1}}$.

Example: Find $\dfrac{dy}{dx}$ when $y = \log_x 10$.

$$x^y = x^{\log_x 10} = 10.$$

Take the derivative:

$$\frac{d\ x^y}{dx} = \frac{d\ (e^{\ln x})^y}{dx} = \frac{d\ e^y \ln x}{dx} = e^{y\ln x}\left(y\cdot\frac{1}{x} + \frac{dy}{dx}\ln x\right)$$
$$= \frac{d10}{dx} = 0.$$

So $\quad e^{y\ln x}\left(y\cdot\dfrac{1}{x} + \dfrac{dy}{dx}\ln x\right) = 0.$

Solve for $\dfrac{dy}{dx}$.

$$e^{y\ln x}\frac{dy}{dx}\ln x = \frac{-e^{y\ln x}y}{x}. \qquad \text{So} \quad \frac{dy}{dx} = \frac{-e^{y\ln x}y}{xe^{y\ln x}\ln x} = \frac{-y}{x\ln x} = \frac{\log_x 10}{x\ln x}.$$

Example: Find the third derivative of 2^x with respect to x.

$y = 2^x$.

$$\frac{dy}{dx} = \frac{d2^x}{dx} = \frac{2(e^{\ln 2})^x}{dx} = \frac{de^{x\ln 2}}{dx} = e^{x\ln 2}(\ln 2) = (e^{\ln 2})^x \ln 2 = 2^x \ln 2.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d2^x\ln 2}{dx} = \ln 2\cdot 2^x\ln 2 = (\ln 2)^2 2^x.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\big(\ln 2)^2 2^x\big) = (\ln 2)^2 2^x \ln 2 = (\ln 2)^3 2^x.$$

Example: If $y = a\cos(\ln x) + b\sin(\ln x)$ show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

$$\begin{aligned}\frac{dy}{dx} &= a(-\sin(\ln x))\frac{1}{x} + b\cos(\ln x)\frac{1}{x}\\ &= -a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1},\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -a\cos(\ln x)\frac{1}{x}x^{-1} + -a\sin(\ln x)(-1)x^{-2} + \ -b\sin(\ln x)\frac{1}{x}x^{-1} + b\cos(\ln x)(-1)x^{-2}\\ &= \frac{-a\cos(\ln x) + a\sin(\ln x) - b\sin(\ln x) - b\cos(\ln x)}{x^2}\\ &= \frac{1}{x^2}\big((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\big).\end{aligned}$$

So

$$\begin{aligned}
LHS &= x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y \\
&= x^2\,\frac{1}{x^2}\big((a-b)\sin(\ln x) - (a+b)\cos(\ln x)\big) \\
&\qquad + x\big(-a\sin(\ln x)x^{-1} + b\cos(\ln x)x^{-1}\big) \\
&\qquad + a\cos(\ln x) + b\sin(\ln x) \\
&= (a-b)\sin(\ln x) - (a+b)\cos(\ln x) \\
&\quad - a\sin(\ln x) + b\cos(\ln x) \\
&\quad + b\sin(\ln x) + a\cos(\ln x) \\
&= 0.
\end{aligned}$$

Example: Find $\frac{dy}{dx}$ when $a\sin(xy) + b\cos\left(\frac{x}{y}\right) = 0$.

Take the derivative:

$$\begin{aligned}
0 &= a\cos(xy)\left(x\frac{dy}{dx} + 1\cdot y\right) + \ -b\sin\left(\frac{x}{y}\right)\left(x(-1)y^{-2}\frac{dy}{dx} + 1\cdot y^{-1}\right) \\
&= a\cos(xy)x\frac{dy}{dx} + a\cos(xy)y + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} - b\sin\left(\frac{x}{y}\right)y^{-1}.
\end{aligned}$$

Solve for $\frac{dy}{dx}$.

$$a\cos(xy)x\frac{dy}{dx} + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}\frac{dy}{dx} = a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}.$$

So

$$\frac{dy}{dx} = \frac{a\cos(xy)y - b\sin\left(\frac{x}{y}\right)y^{-1}}{a\cos(xy)x + b\sin\left(\frac{x}{y}\right)\frac{x}{y^2}}$$

$$= \frac{a\cos(xy)y^3 - b\sin\left(\frac{x}{y}\right)y}{a\cos(xy)xy^2 + b\sin\left(\frac{x}{y}\right)x}$$

Example: Find $\frac{dy}{dx}$ when $y = \tan^{-1}\left(\frac{a}{x}\right)\cdot\cot^{-1}\left(\frac{x}{a}\right)$.

$$\frac{dy}{dx} = \tan^{-1}\left(\frac{a}{x}\right)\left(\frac{-1}{1+\left(\frac{x}{a}\right)^2}\right)\frac{1}{a} + \frac{1}{1+\left(\frac{x}{a}\right)^2}(-1)ax^{-2}\cot^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)}{a+\frac{x^2}{a}} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \frac{-\tan^{-1}\left(\frac{a}{x}\right)a}{a^2+x^2} + \frac{-\cot^{-1}\left(\frac{x}{a}\right)a}{x^2+a^2}$$

$$= \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \cot^{-1}\left(\frac{x}{a}\right)\right).$$

If $\frac{a}{x} = \tan z$ then $\frac{x}{a} = \cot z$ and $z = \tan^{-1}\left(\frac{a}{x}\right) = \cot^{-1}\left(\frac{x}{a}\right)$.

So

$$\frac{dy}{dx} = \left(\frac{-a}{a^2+x^2}\right)\left(\tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{a}{x}\right)\right) = \frac{-2a\tan^{-1}\left(\frac{a}{x}\right)}{a^2+x^2}.$$