The basic trig identities

Arun Ram
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706 USA
ram@math.wisc.edu

Version: September 6, 2006

Define

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Example: Explain why $e^{ix} = \cos x + i \sin x$, if $i = \sqrt{-1}$.

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots, \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \cdots, \\
&= 1 + ix + \frac{i^2x^2}{2!} + \frac{i \cdot i^2x^3}{3!} + \frac{((i^2)^2x^4}{4!} + \frac{i \cdot (i^2)^2x^5}{5!} + \frac{(i^2)^3x^6}{6!} + \frac{i \cdot (i^2)^3x^7}{7!} + \cdots, \\
&= 1 + ix + \frac{(-1)x^2}{2!} + \frac{i \cdot (-1)x^3}{3!} + \frac{(-1)^2x^4}{4!} + \frac{i \cdot (-1)^2x^5}{5!} + \frac{(-1)^3x^6}{6!} + \frac{i \cdot (-1)^3x^7}{7!} + \cdots, \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots, \\
&= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) \\
&= \cos x + i \sin x.
\end{aligned}$$

Example: Explain why $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$.

$$\begin{aligned}\cos(-x) &= 1 - \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} - \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} - \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} - \cdots, \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots, \\ &= \cos x,\end{aligned}$$

and

$$\begin{aligned}\sin(-x) &= (-x) - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} - \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} - \cdots, \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \frac{x^{11}}{11!} - \frac{x^{13}}{13!} + \cdots, \\ &= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots\right), \\ &= -\sin x.\end{aligned}$$

Example: Explain why $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}1 = e^0 &= e^{ix+(-ix)} \\ &= e^{ix}e^{-ix} = e^{ix}e^{i(-x)} \\ &= (\cos x + i\sin x)(\cos(-x) + i\sin(-x)) \\ &= (\cos x + i\sin x)(\cos x + i(-\sin x)) \\ &= \cos^2 x - i\sin x\cos x + i\sin x\cos x - i^2\sin^2 x \\ &= \cos^2 x - (-1)\sin^2 x \\ &= \cos^2 x + \sin^2 x.\end{aligned}$$

Example: Explain why $\cos(x+y) = \cos x\cos y - \sin x\sin y,$ and $\sin(x+y) = \sin x\cos y + \cos x\sin y.$

$$\begin{aligned}\cos(x+y) + i\sin(x+y) &= e^{i(x+y)} \\ &= e^{ix+iy} = e^{ix}e^{iy} \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= \cos x\cos y + i\cos x\sin y + i\sin x\cos y + i^2\sin x\sin y \\ &= \big(\cos x\cos y + (-1)\sin x\sin y\big) + i\big(\cos x\sin y + \sin x\cos y\big).\end{aligned}$$

Comparing terms on each side gives

$$\cos(x+y) = \cos x\cos y - \sin x\sin y, \qquad \text{and}$$
$$\sin(x+y) = \sin x\cos y + \cos x\sin y.$$

Define

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots, \\
\sinh x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \cdots, \\
\cosh x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \cdots,
\end{aligned}
$$

and

$$
\tanh x = \frac{\sinh x}{\cosh x}, \quad \mathrm{coth} x = \frac{1}{\tanh x}, \quad \mathrm{sech} x = \frac{1}{\cosh x}, \quad \mathrm{csch} x = \frac{1}{\sinh x}.
$$

Example: Explain why $e^x = \cosh x + \sinh x$.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\
&= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) \\
&= \cosh x + \sinh x.
\end{aligned}
$$

Example: Explain why $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$.

$$
\begin{aligned}
\cosh(-x) &= 1 + \frac{(-x)^2}{2!} + \frac{(-x)^4}{4!} + \frac{(-x)^6}{6!} + \frac{(-x)^8}{8!} + \frac{(-x)^{10}}{10!} + \frac{(-x)^{12}}{12!} + \cdots \\
&= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots \\
&= \cosh x,
\end{aligned}
$$

and

$$
\begin{aligned}
\sinh(-x) &= (-x) + \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} + \frac{(-x)^7}{7!} + \frac{(-x)^9}{9!} + \frac{(-x)^{11}}{11!} + \frac{(-x)^{13}}{13!} + \cdots \\
&= -x - \frac{x^3}{3!} - \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} - \frac{x^{11}}{11!} - \frac{x^{13}}{13!} - \cdots \\
&= -\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots\right) \\
&= -\sinh x.
\end{aligned}
$$

Example: Explain why $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

$$\begin{aligned}
\tfrac{1}{2}(e^x+e^{-x}) &= \tfrac{1}{2}\Big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots \\
&\qquad +1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\frac{(-x)^5}{5!}+\frac{(-x)^6}{6!}+\frac{(-x)^7}{7!}+\cdots\Big) \\
&= \tfrac{1}{2}\Big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots \\
&\qquad +1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}-\frac{x^7}{7!}+\cdots\Big) \\
&= 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\frac{x^8}{8!}+\cdots \\
&= \cosh x.
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(e^x-e^{-x}) &= \tfrac{1}{2}\Big(\big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots\big) \\
&\qquad -\big(1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}+\frac{(-x)^5}{5!}+\frac{(-x)^6}{6!}+\frac{(-x)^7}{7!}+\cdots\big)\Big) \\
&= \tfrac{1}{2}\Big(\big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots\big) \\
&\qquad -\big(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}-\frac{x^7}{7!}+\cdots\big)\Big) \\
&= \tfrac{1}{2}\Big(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots \\
&\qquad -1+x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}-\frac{x^6}{6!}+\frac{x^7}{7!}-\cdots\Big) \\
&= x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\cdots \\
&= \sinh x.
\end{aligned}$$

Example: Explain why $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}
1 = e^0 &= e^{x+(-x)} \\
&= e^x e^{-x} \\
&= (\cosh x + \sinh x)(\cosh(-x) + \sinh(-x)) \\
&= (\cosh x + \sinh x)(\cosh x - \sinh x)) \\
&= \cosh^2 x - \sinh x \cosh x + \sinh x \cosh x - \sinh^2 x \\
&= \cosh^2 x - \sinh^2 x.
\end{aligned}$$

Example: Explain why $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$ and
$\sinh(x+y) = 2 \sinh x \cosh y.$

$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y + e^{-x} e^y + e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&\quad + \frac{e^x e^y - e^{-x} e^y - e^x e^{-y} + e^{-x} e^{-y}}{4} \\
&= \frac{2e^x e^y + 2e^{-x} e^{-y}}{4} \\
&= \frac{e^{(x+y)} + e^{-(x+y)}}{2} \\
&= \cosh(x+y).
\end{aligned}$$

and

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\
&= \frac{e^x e^y - e^{-x} e^y + e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&\quad + \frac{e^x e^y + e^{-x} e^y - e^x e^{-y} - e^{-x} e^{-y}}{4} \\
&= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} \\
&= \sinh(x+y).
\end{aligned}$$