Math 541
Modern Algebra
A first course in Abstract Algebra
Lecturer: Arun Ram

Fall 2007

University of Wisconsin-Madison
Mathematics Department

Homework 1: Due September 12, 2007

1. Define set, emptyset, subset, equal sets, union of sets, intersection of sets, and product of sets and give examples.

2. Define function, injective, surjective, bijective, composition of funtions, identity function, and inverse function and give examples.

3. Define relation, reflexive relation, symmetric relation, transitive relation, equivalence relation, partial order, and poset and give examples.

4. Define operation, commutative, and associative and give examples.

5. Explain what the positive integers, the nonnegative integers, the integers, the rational numbers, the real numbers and the complex numbers are and why each of these numbers systems is needed.

6. Show that the empty set is a subset of every set.

7. De Morgan's Laws: Let A, B, and C be sets. Show that

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$

   (b) $A \cup B = B \cup A$

   (c) $A \cup \emptyset = A$

   (d) $(A \cap B) \cap C = A \cap (B \cap C)$

   (e) $A \cap B = B \cap A$

   (f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8. Prove Theorem 1.1 on p. 5 of the text. Theorem 1.1 on p. 5 of the text reads:

Theorem. *Let A, B, and C be sets. Then*

(i) $A \cup A = A = A \cap A$.
(ii) $A \cup B = B \cup A$; $A \cap B = B \cap A$.
(iii) $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$.
(iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(v) $A \cup (A \cap B) = A = A \cap (A \cup B)$.

9. Define complement of a set. Restate Theorem 1.2 (from p. 6 of the text) in terms of complements and prove it. Theorem 1.2 on p. 6 reads:

Theorem (DeMorgan's rules). *Let A, B, and X be sets. Then*

(i) $X - (X - A) = X \cap A$.
(ii) $X - (A \cup B) = (X - A) \cap (X - B)$.
(iii) $X - (A \cap B) = (X - A) \cup (X - B)$.

10. Let S, T, and U be sets and let $f : S \to T$ and $g : T \to U$ be functions.

(a) If f and g are injective then $g \circ f$ is injective.

(b) If f and g are surjective then $g \circ f$ is surjective.

(a) If f and g are bijective then $g \circ f$ is bijective.

11. Let $f : S \to T$ be a function and let $U \subseteq S$. The *image* of U under f is the subset of T given by
$$f(U) = \{f(u) \mid u \in U\}.$$
Let $f : S \to T$ be a function. The *image* of f is the subset of T given by
$$\mathrm{im}(f) = \{f(s) \mid s \in S\}.$$
Note that $\mathrm{im}\, f = f(S)$.

Let $f : S \to T$ be a function and let $V \subseteq T$. The *inverse image* of V under f is the subset of S given by
$$f^{-1}(V) = \{s \in S \mid f(s) \in V\}.$$

Let $f : S \to T$ be a function and let $t \in T$. The *fiber* of f over t is the subset of S given by
$$f^{-1}(t) = \{s \in S \mid f(s) = t\}.$$
Note that $f^{-1}(t) = f^{-1}(\{t\})$.

Let $f : S \to T$ be a function. Show that the set $F = \{f^{-1}(t) \mid t \in T\}$ of nonempty fibers of the map f is a partition of S.

12. (a) Let $f : S \to T$ be a function. Define
$$\begin{array}{rccc} f' : & S & \longrightarrow & \mathrm{im}(f) \\ & s & \mapsto & f(s) \end{array}$$

Show that the map f' is well defined and surjective.

(b) Let $f : S \to T$ be a function and let $F = \{f^{-1}(t) \mid t \in T\}$ be the set of nonempty fibers of f. Define

$$\begin{array}{cccc} \widehat{f}: & F & \longrightarrow & T \\ & f^{-1} & \mapsto & t \end{array}$$

Show that the map $\widehat{f}$ is well defined and injective.

(c) Let $f : S \to T$ be a function and let $F = \{f^{-1}(t) \mid t \in T\}$ be the set of nonempty fibers of f. Define

$$\begin{array}{cccc} \widehat{f}' & F & \longrightarrow & \operatorname{im} f \\ & f^{-1}(t) & \mapsto & t \end{array}$$

Show that the map $\widehat{f}'$ is well defined and bijective.

13. Let S be a set. The *power set* of S, 2^S, is the set of all subsets of S.

    Let S be a set and let $\{0,1\}^S$ be the set of all functions $f : S \to \{0,1\}$. Given a subset $T \subseteq S$ define a function

$$f_T : S \to \{0,1\} \qquad \text{by} \qquad f_T = \begin{cases} 0, & \text{if } s \notin T, \\ 1, & \text{if } s \in T. \end{cases}$$

    Show that the map

$$\begin{array}{cccc} \psi: & 2^S & \longrightarrow & \{0.1\}^S \\ & T & \mapsto & f_T \end{array} \qquad \text{is a bijection.}$$

14. Let $\circ \ : S \times S \to S$ be an associative operation on a set S. An *identity* for $\circ$ is an element $e \in S$ such that

$$e \circ s = s \circ e, \qquad \text{for all} \quad s \in S.$$

    Let e be an identity for an associative operation compfn on a set S. Let $s \in S$. A *left inverse* for s is an element $t' \in S$ such that $s \circ t' = e$. An *inverse* for s is an element $s^{-1} \in S$ such that $s \circ s^{-1} = s^{-1} \circ s = e$.

    (a) Let $\circ$ be an operation on a set S. Show that if S contains an identity for $\circ$ then it is unique.

    (b) Let e be an identity for an associative operation $\circ$ on a set S. Let $s \in S$. Show that if s has an inverse then it is unique.

15. Let S and T be sets and let ι_S and ι_T be the identity maps on S and T, respectively.

    (a) Show that for any function $f : S \to T$,

$$\iota_T \circ f = f \qquad \text{and} \qquad f \circ \iota_S = f.$$

(b) Let $f : S \to T$ be a function. Show that if an inverse function to f exists then it is unique.

16. Define partition and equivalence class and state and prove a theorem which shows that these two definitions contain the same information.