University of Wisconsin-Madison
Mathematics Department

Math 541
Modern Algebra
A first course in Abstract Algebra
Lecturer: Arun Ram

Fall 2007

Homework 11: Due November 21, 2007

To grade: 5, 10, 11, 13, 14, 17.

1. Let S be a subset of a group G. Define the subgroup generated by S.

2. Let S be a subset of a ring R. Define the ideal generated by S.

3. Let S be a subset of a module M. Define the submodule generated by S.

4. Let S be a subset of a vector space V. Define the subspace generated by S.

5. Let S be a subset of a vector space V. Show that span(S) is equal to the set of linear combinations of elements of S.

6. Show that the intersection of two subgroups of a group G is a subgroup of G.

7. Give an example to show that the union of two subgroups of G is not necessarily a subgroup of G.

8. Let G be a group and let S be a subset of G. Let $\mathcal{H}$ be the set of subgroups H of G such that $S \subseteq H$. Define
$$H_S = \cap_{H \in \mathcal{H}} H$$
   1. Show that H_S is a subgroup of G.
   2. Show that $S \subseteq H_S$.
   3. Show that if H is a subgroup of G and $S \subseteq H$ then $H_S \subseteq H$.

   Conclude that $H_S = \langle S \rangle$.

9. Determine the subgroup lattice of the dihedral group D_4. The group D_4 acts on its subgroups by conjugation. Determine the stabilizer and the orbit of each subgroup.

10. Determine the subgroup lattice of the quaternion group Q. The group Q acts on its subgroups by conjugation. Determine the stabilizer and the orbit of each subgroup.

11. Determine the subgroup lattice of the dihedral group D_5. The group D_5 acts on its subgroups by conjugation. Determine the stabilizer and the orbit of each subgroup.

12. The dihedral group D_4 acts on its elements by conjugation. Determine the stabilizer and the orbit of each element. Determine the conjugacy classes of D_4, the centralizer of each element, and determine the center of D_4.

13. The quaternion group Q acts on its elements by conjugation. Determine the stabilizer and the orbit of each element. Determine the conjugacy classes of Q, the centralizer of each element, and determine the center of Q.

14. The dihedral group D_5 acts on its elements by conjugation. Determine the stabilizer and the orbit of each element. Determine the conjugacy classes of D_5, the centralizer of each element, and determine the center of D_5.

15. Determine the subgroup lattice of the quaternion group Q. The group Q acts on its subgroups by conjugation. Determine the stabilizer and the orbit of each subgroup.

16. Let A be a matrix. Explain how to use A to produce a module for the ring $\mathbb{C}[x]$.

17. Use the matrix

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 6 & 3 \\ 6 & -12 & -6 \end{pmatrix}$$

to define a $\mathbb{C}[x]$-module on the vector space V with basis b_1, b_2, b_3. Compute $(x^2 + 2x + 1)b_2$.

18. Find all submodules of the module in Problem 17.

19. Let V be the vector space with basis b_1, b_2, b_3. The matrix

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -3 & 6 & 3 \\ 6 & -12 & -6 \end{pmatrix}$$

defines a linear transformation $f : V \to V$. Find $\ker f$.