THE UNIVERSITY of WISCONSIN MADISON

University of Wisconsin-Madison
Mathematics Department

Math 541
Modern Algebra
A first course in Abstract Algebra
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Fall 2007

Homework 14: Due December 13, 2007

To grade: 13, 14, 15, 16.

1. Define the following terms.
    - vector space
    - subspace
    - span(S)
    - linear combination
    - linearly independent
    - basis
    - linear transformation
    - kernel (of a linear transformation)
    - image (of a linear transformation)
    - eigenvector with eigenvalue λ
2. Define $M_n(\mathbb{F})$ and appropriate operations and prove that it is a ring.
3. Define $M_{n\times m}(\mathbb{F})$ and appropriate operations/actions and prove that it is a vector space.
4. Define $\mathbb{F}^n$ and prove that it is a vector space.
5. Let V be a vector space with basis $\{s_1, \cdots s_n\}$. Prove that $V \simeq \mathbb{F}^n$.
6. Let V and W be vector spaces and let $\mathrm{Hom}(V, W)$ be the set of linear transformations from V to W. Define appropriate operations/actions and prove that $\mathrm{Hom}(V, W)$ is a vector space.
7. Let V be a vector space. Let $\mathrm{End}(V)$ be the set of linear transformations from V to V. Define appropriate operations and prove that $\mathrm{End}(V)$ is a ring.
8. Let V be a vector space with basis $\{s_1, \cdots s_n\}$. Prove that $\mathrm{End}(V) \simeq M_n(\mathbb{F})$ (as rings).
9. Let W be a vector space with basis $\{t_1, \cdots t_m\}$ and let V be a vector space with basis $\{s_1, \cdots s_n\}$. Prove that $\mathrm{Hom}(V, W) \simeq M_{n\times m}(\mathbb{F})$ (as vector spaces).

10. Let $f : V \to W$ be a linear transformation. Show that f is injective if and only if $\ker\ f = 0$.

11. Let V be a finite dimensional vector space and let $f : V \to V$ be a linear transformation. Show that f is invertible if and only if $\ker\ f = 0$.

12. Let V be a finite dimensional vector space and let $f : V \to V$ be a linear transformation. Let $\lambda \in \mathbb{F}$. Define the λ-eigenspace V_λ of f. Find a linear transformation $h : V \to V$ such that $\ker\ h = V_\lambda$.

13. Define the determinant.
(a) Prove that the determinant is a monoid homomorphism.
(b) Prove that the determinant is a group homomorphism.
(c) Prove that the determinant is a ring homomorphism.

14. Write a formula for $\det(A)$ which corresponds to Laplace expansion down the first column. Prove this formula. Interpret this formula in terms of cosets.

15. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if $\det(A)$ is invertible.

16. Let W be a vector space with basis $\{t_1, \cdots t_m\}$ and let V be a vector space with basis $\{s_1, \cdots s_n\}$. Let $\varphi_S : V \to \mathbb{F}^n$, $\varphi_T : W \to \mathbb{F}^m$ and $\Phi : \mathrm{Hom}(V, W) \to M_{n \times m}(\mathbb{F})$ be the corresponding isomorphisms (see problem 5 and problem 13). Let $f : V \to V$ be a linear transformation. Prove that
(a) $\varphi_S(\ker\ f)$ is equal to the null space of $\Phi(f)$.
(b) $\varphi_T(\mathrm{im}\ f)$ is equal to the column space of $\Phi(f)$.
If you give proof machine definitions of *null space* and *column space* before beginning these proofs this problem is not difficult but if you don't it is impossible (and probably doesn't make any sense).