Math 541
Modern Algebra
A first course in Abstract Algebra
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Fall 2007

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Homework 6: Due October 17, 2007

To grade: 6, 10, 17.

1. Define group, subgroup, coset, G/H and normal subgroup.
2. Make a list of the groups with $\le$ 10 elements their subgroups and the corresponding G/H.
3. Show that $5\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ and explicitly determine the cosets of $5\mathbb{Z}$ in $\mathbb{Z}$.
4. Let G be a group, H a subgroup and let $g \in G$ and $h \in H$. Show that $gH = ghH$.
5. Let G be a group, H a subgroup and let $x, g \in G$. Show that $x \in gH$ if and only if $gH = xH$.
6. Let G be a group and H a subgroup. Show that G/H is a partition of G.
7. Let G be a group, H a subgroup and let $g_1, g_2 \in G$. Show that $\mathrm{Card}(g_1H) = \mathrm{Card}(g_2H)$.
8. Let H be a subgroup of a group G. Show that $\mathrm{Card}(G) = \mathrm{Card}(G/H)\mathrm{Card}(H)$.
9. Define integral domain and field of fractions and give examples.
10. Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that the operation $+ : \mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F}$ is well defined.
11. Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that the operation $\cdot : \mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F}$ is well defined.
12. Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that $\mathbb{F}$ is a group.
13. Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Define abelian group and show that $\mathbb{F}$ is an abelian group.

14. Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that $\mathbb{F}$ is a ring.

15. Let R be an integral domain and let $\mathbb{F}$ be its field of fractions. Show that $\mathbb{F}$ is a field.

16. Let H be a subgroup of a group G. Show that if the operation on G/H given by
$$(g_1H)(g_2H) = g_1g_2H$$
is well defined then H is a normal subgroup of G

17. Let H be a subgroup of a group G. Show that if H is a normal subgroup of G then the operation on G/H given by
$$(g_1H)(g_2H) = g_1g_2H$$
is well defined.

18. Let H be a normal subgroup of a group G. Show that G/H with operation given by
$$(g_1H)(g_2H) = g_1g_2H$$
is a group.

19. Show that *every* subgroup of an abelian group is normal.

20. Let $f : G \longrightarrow H$ be a group homomorphism. Show that $\ker f$ is a subgroup of G.

21. Let $f : G \longrightarrow H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of G.

22. Let $f : G \longrightarrow H$ be a group homomorphism. Show that $\operatorname{im} f$ is a subgroup of H.

23. Let $f : G \longrightarrow H$ be a group homomorphism. Show that $\operatorname{im} f$ is a normal subgroup of H.