Theorems 2.20-2.22: Finite dimensional vector spaces.

Theorem 2.20 Let X be a finite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. Let

$$B = \{u_1, u_2, \ldots, u_N\} \text{ be a basis of } X.$$

(a) X is complete.

(b) The map $\Lambda : \mathbb{K}^N \longrightarrow X$ given by

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_N) = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_N u_N$$

is linear, bounded, bijective and has bounded inverse.

Theorem 2.22 Let X be a normed vector space. X is finite dimensional if and only if

$$\overline{B(0,1)} = \{x \in X \mid \|x\| \leq 1\} \quad \text{is compact.}$$

Theorem A.3 Let S be a metric space. The following are equivalent:

(a) S is compact.

(b) S is precompact and complete.

(c) From every sequence $(x_1, x_2, \ldots)$ of points in S one can extract a subsequence converging to some limit point $\exists\, x \in S$.

Dualspaces in the world of functional analysis

Let X be a vectorspace over $\mathbb{K}$.

In normal linear algebra, the dual vector space to X is

$$X^* = \{\varphi : X \to \mathbb{K} \mid \varphi \text{ is a linear operator}\}$$

with

$$(\varphi_1+\varphi_2)(x) = \varphi_1(x)+\varphi_2(x) \quad \text{and} \quad (c\varphi)(x) = c\,\varphi(x).$$

Let X be a normed vectorsspace with norm $\|\cdot\|$.

The dual vector space to X, in the world of functional analysis, is

$$X^* = B(X,\mathbb{K}) = \left\{\varphi : X \to \mathbb{K} \;\middle|\; \begin{array}{l}\varphi \text{ is a linear operator}\\ \text{and } \|\varphi\| < \infty\end{array}\right\}.$$

There is an inclusion (injective function)

$$\begin{aligned} \iota : X &\hookrightarrow (X^*)^* \\ x &\longmapsto \Lambda_x : X^* \to \mathbb{K} \\ & \qquad\quad \varphi \mapsto \varphi(x) \end{aligned}$$

The vectorspace X is reflexive if ι is surjective.

HW: Show that $L^1(\Omega)$, $L^\infty(\Omega)$, ℓ^1 and ℓ^∞ are not reflexive.

Extension theorems

Theorem 2.29 Let X be a vector space over $\mathbb{R}$.
Let $p: X \to \mathbb{R}$ be a function such that

(a) if $x, y \in X$ then $p(x+y) \le p(x) + p(y)$

(b) if $x \in X$ and $t \in \mathbb{R}_{\ge 0}$ then $p(tx) = t\,p(x)$.

Let $V \subseteq X$ be a subspace and
$f: V \to \mathbb{R}$ a linear functional such that
if $x \in V$ then $f(x) \le p(x)$.

Then there exists a linear functional $F: X \to \mathbb{R}$ such that

(a) if $x \in V$ then $F(x) = f(x)$

(b) If $x \in X$ then $-p(-x) \le F(x) \le p(x)$.

Inductive Construction: Assume $V \neq X$ and let $x_0 \in X$ with $x_0 \in V$ and $x_0 \neq$
Let

$$\beta = \sup\{f(x) - p(x - x_0) \mid x \in V\} \text{ and}$$

$$f(x + tx_0) = f(x) + \beta t, \text{ for } x \in V, t \in \mathbb{R}.$$

Let $f: V_0 \to \mathbb{R}$ with $V_0 = \{x + tx_0 \mid x \in V, t \in \mathbb{R}\} \supsetneq V$.

Theorem 2.30 Let $(X, \|\cdot\|)$ be a normed vector space, $V \subseteq X$ a subspace
Let $f: V \to \mathbb{K}$ be a bounded linear functional.
Then there exists a linear functional $F: X \to \mathbb{K}$ such that

(a) If $v \in V$ then $F(v) = f(v)$,

(b) $\|F\| = \|f\|$.

Convergence and weak convergence

Let $\mathbb{K}$ be the field of real numbers of the field of complex numbers.

Let $\mathbb{Y}$ be a Banach space over $\mathbb{K}$.

The dual space to $\mathbb{Y}$ is

$$\mathbb{Y}^* = \left\{ \varphi : \mathbb{Y} \to \mathbb{K} \;\middle|\; \begin{matrix} \varphi \text{ is a linear operator} \\ \text{and } \|\varphi\| < \infty \end{matrix} \right\}$$

where $\|\varphi\| = \sup\{ \|\varphi(y)\| \mid y \in \overline{B(0,1)} \}$.

A sequence $(y_1, y_2, \ldots)$ in $\mathbb{Y}$ <u>converges</u> if there exists $y \in \mathbb{Y}$ such that

$$\lim_{n\to\infty} \|y_n - y\| = 0.$$

A sequence $(y_1, y_2, \ldots)$ in $\mathbb{Y}$ <u>weakly converges</u> if there exists $y \in \mathbb{Y}$ such that

if $\varphi \in \mathbb{Y}^*$ then $\displaystyle\lim_{n\to\infty} (\varphi(y_n) - \varphi(y)) = 0$.

Let $\mathbb{K}$ be the real numbers or the complex numbers.
Let X be a Banach space over $\mathbb{K}$.
The norm on

5.7

$$X^* = B(X, \mathbb{K}) = \left\{ \varphi : X \to \mathbb{K} \,\middle|\, \begin{array}{l} \varphi \text{ is a linear operator} \\ \text{and } \|\varphi\| < \infty \end{array} \right\}$$

is given by

$$\|\varphi\| = \sup\{ \|\lambda x\| \mid x \in \overline{B(0,1)} \}.$$

A sequence $(\varphi_1, \varphi_2, \ldots)$ in X^* <u>converges</u> if there exists $\varphi \in X^*$ such that

$$\lim_{n \to \infty} \|\varphi_n - \varphi\| = 0.$$

A sequence $(\varphi_1, \varphi_2, \ldots)$ in X^* ~~__~~ <u>weakly converges</u> ~~if it satisfies~~: if there exists $\varphi \in X^*$ such that

if $\Lambda \in (X^*)^*$ then $\lim_{n \to \infty} (\Lambda(\varphi_n) - \Lambda(\varphi)) = 0.$

A sequence $(\varphi_1, \varphi_2, \ldots)$ in X^* <u>weak $*$ converges</u> if there exists $\varphi \in X^*$ such that

if $x \in X$ then $\lim_{n \to \infty} (\varphi_n(x) - \varphi(x)) = 0.$

~~Com~~ <u>Weak $*$ compactness</u> (Banach-Alaoglu theorem). Let X be a Banach space. If $(\varphi_1, \varphi_2, \ldots)$ is a bounded sequence in X^* then there exists a weak$*$ convergent subsequence of $(\varphi_1, \varphi_2, \ldots)$.