§3.3 Ascoli's compactness theorem.

HW: Show that the sequence $f_1, f_2, \ldots$ in $C([0,1])$ given by $f_k = x^k$, is bounded but does not contain any uniformly convergent subsequence.

HW: Show that the sequence $f_1, f_2, \ldots$ in $C([0,1])$ given by $f_k = \sin kx$, is bounded but does not contain any uniformly convergent subsequence.

Let E be a metric space and $\mathcal{F} \subseteq C(E)$.
The subset $\mathcal{F}$ is equicontinuous if $\mathcal{F}$ satisfies:
if $x \in E$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $f \in \mathcal{F}$ and
if $d(y,x) < \delta$ then $|f(y) - f(x)| < \varepsilon$.

The subset $\mathcal{F}$ is uniformly equicontinuous if $\mathcal{F}$ satisfies
if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that
if $f \in \mathcal{F}$ and $x, y \in E$ and $d(y,x) < \delta$
then $|f(x) - f(y)| < \varepsilon$.

Lemma 3.11 Let E be a compact metric space and let $\mathcal{F} \subseteq C(E)$, with $\mathcal{F}$ equicontinuous. Then $\mathcal{F}$ is uniformly equicontinuous.

HW Let E be a metric space and let $\mathcal{F} \subseteq C(E)$

(a) Define $\mathcal{F}$ is relatively compact.

(b) Define $\mathcal{F}$ is precompact

(c) Show that if E is compact then the following are equivalent

(1) $\mathcal{F}$ is relatively compact

(2) $\mathcal{F}$ is precompact

(3) If $f_1, f_2, \ldots$ is a sequence in $\mathcal{F}$ then there exists a subsequence converging uniformly to a function $f \in C(E)$.

Theorem 3.12 (Ascoli). Let E be a compact metric space. Let $\mathcal{F} \subseteq C(E)$ be equicontinuous such that if $x \in E$ then $\sup\{ |f(x)| \mid f \in \mathcal{F}\} < \infty$,

Then $\mathcal{F}$ is relatively compact

Corollary 3.13 Let E be a compact metric space. Let $f_1, f_2, \ldots$ be a sequence in $C(E, \mathbb{R}^n)$ such that

(i) $\{f_1, f_2, \ldots\}$ is equicontinuous, and

(ii) if $x \in E$ then $\sup\{ |f_k(x)| \mid k \in \mathbb{Z}_{>0}\} < \infty$.

Then $f_1, f_2, \ldots$ has a uniformly convergent subsequence

§3.4 Spaces of Hölder continuous functions.

Let $\Omega \subseteq \mathbb{R}^n$ be open. For $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ let

$$D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f.$$

Let $\gamma \in (0,1]_{\mathbb{R}}$. The space $C^{0,\gamma}(\Omega)$ of <u>bounded Hölder continuous functions with exponent</u> γ is

is the set of functions $f: \Omega \to \mathbb{R}$ such that f is bounded and there exists ~~a constan~~ $C \in \mathbb{R}_{>0}$ such that

$$\text{if } x, y \in \Omega \text{ then } |f(x) - f(y)| \leq C|x-y|^\gamma, \qquad (*)$$

and $C^{0,\gamma}(\Omega)$ has norm given by

$$\|f\|_{C^{0,\gamma}(\Omega)} = \sup\{|f(x)| \mid x \in \Omega\} + \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|^\gamma} \;\middle|\; x, y \in \Omega \text{ and } x \neq y \right\}.$$

Let $k \in \mathbb{Z}_{>0}$.

The space $C^{k,\gamma}(\Omega)$ is

$$C^{k,\gamma}(\Omega) = \left\{ f: \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} \text{if } (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0} \text{ and } \alpha_1 + \cdots + \alpha_n \leq k \\ \text{then } D^\alpha f \text{ satisfies } (*) \end{array} \right\},$$

with norm given by

$$\|f\|_{C^{k,\gamma}(\Omega)} = \sum_{|\alpha| \leq k} \sup\{|D^\alpha f(x)| \mid x \in \Omega\} + \sum_{|\alpha| = k} \sup\left\{ \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\gamma} \;\middle|\; \begin{array}{l} x, y \in \Omega \\ \text{and } x \neq y \end{array} \right\}.$$

Theorem (3.14) (Hölder spaces are complete).

Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{Z}_{\geq 0}$ and $\gamma \in (0,1]_{\mathbb{R}}$.

Then $C^{k,\gamma}(\Omega)$ is a Banach space.

Remark We should really define

$$C^{k,\gamma}(\Omega) = \{f : \Omega \to \mathbb{R} \mid \|f\|_{C^{k,\gamma}(\Omega)} < \infty\}.$$