§5.5.1 Let H be a Hilbert space.

An <u>orthonormal basis of H</u> is a sequence $S = \{e_1, e_2, \dots\}$ in H such that if $i, j \in \mathbb{Z}_{>0}$ then

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad \text{and} \quad \overline{\mathrm{span}(S)} = H.$$

<u>HW</u>: Assume H is a Hilbert space and S is an orthonormal basis of H. Let $V = \overline{\mathrm{span}(S)}$. Show that if $x \in H$ then

$$P_V(x) = x, \quad \sum_{k \in \mathbb{Z}_{>0}} |\langle x, e_k \rangle|^2 = \|x\|^2, \quad \text{and} \quad x = \sum_{k \in \mathbb{Z}_{>0}} \langle x, e_k \rangle e_k.$$

<u>Example</u>: <u>Fourier series</u>

Let $H = L^2([-\pi, \pi]; \mathbb{C})$ with $\langle \, , \rangle : H \times H \to \mathbb{C}$ given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx.$$

For $n \in \mathbb{Z}$ let

$$\varphi_n : [-\pi, \pi] \to \mathbb{C} \text{ be given by } \varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

(a) Show that $S = \{\varphi_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis of H.

(b) Let $f \in L^2([-\pi, \pi]; \mathbb{C})$. Define

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} \, dy, \quad \text{for } k \in \mathbb{Z}.$$

Show that $\displaystyle\lim_{N \to \infty} \int_{-\pi}^{\pi} \left| f(x) - \sum_{k=-N}^{N} c_k e^{ikx} \right|^2 dx = 0.$

§5.6 Let H be a Hilbert space over $\mathbb{R}$.

A linear operator $A: H \to H$ is strictly positive definite, or strictly monotone, if there exists $\beta \in \mathbb{R}_{>0}$ such that

$$\text{if } u \in H \text{ then } \langle Au, u\rangle \geq \beta \langle u, u\rangle.$$

A continuous bilinear functional on H is a function $B: H \otimes H \to \mathbb{R}$ such that

(a) if $a_1, a_2 \in \mathbb{R}$ and $u_1, u_2, u_3 \in H$ then

$$B((a_1u_1 + a_2u_2)\otimes u_3) = a_1 B(u_1 \otimes u_3) + a_2 B(u_2 \otimes u_3),$$

$$B(u_1 \otimes (a_1u_2 + u_2u_3)) = a_1 B(u_1 \otimes u_2) + a_2 B(u_1 \otimes u_3),$$

(b) There exists $C \in \mathbb{R}_{>0}$ such that

$$\text{if } u_1, u_2 \in H \text{ then } |B(u_1 \otimes u_2)| \leq C\|u_1\|\|u_2\|$$

Theorem 5.12 and 5.13 Let H be a Hilbert space over $\mathbb{R}$. Let $\beta \in \mathbb{R}_{>0}$.

(a) Let $A: H \to H$ be a bounded linear operator such that

$$\text{if } u \in H \text{ then } \langle Au, u\rangle \geq \beta \langle u, u\rangle.$$

Then the inverse function

$$A^{-1}: H \to H \text{ exists and } \|A^{-1}\| \leq \frac{1}{\beta}.$$

(b) Let $\beta \in \mathbb{R}_{>0}$. Let $B: H \otimes H \to \mathbb{R}$ be a continuous bilinear functional such that

if $u \in H$ then $B(u \otimes u) \geq \beta \langle u, u \rangle$.

If $f \in H$ then there is a unique $Df \in H$ such that

if $v \in H$ then $B((Df) \otimes v) = \langle f, v \rangle$

and the function $D: H \to H$ satisfies

if $f \in H$ then $\|Df\| \leq \frac{1}{\beta} \|f\|$.

§5.7 Convergence in a Hilbert space

Theorems 5.14 and 5.15: Let H be a Hilbert space.

(a) A weakly convergent sequence in H is bounded.

(b) A bounded sequence in H has a weakly convergent subsequence.

(c) Let $x \in H$. If $\begin{matrix} \vec{x}: \mathbb{Z}_{>0} \to H \\ n \mapsto x_n \end{matrix}$ is a sequence in H which weakly converges to x and $\Lambda: H \to H$ is a compact operator then the sequence $(\Lambda x_1, \Lambda x_2, \ldots)$ converges strongly to Λx.

Example 5.16:

Let $H = L^2([0,1])$ and define $f_1, f_2, \ldots$ on H by

$$f_n(x) = \sin^2 nx, \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Let $f : [0,1] \to \mathbb{R}$ be the constant function,

$$f(x) = \tfrac{1}{2}.$$

(a) Show that the sequence $f_1, f_2, \ldots$ weakly converges to f.

(b) Show that $f_1, f_2, \ldots$ does not strongly converge to f by showing that

$$\lim_{n \to \infty} \| f_n - f \|_{L^2}^2 = \frac{1}{8}.$$

(c) Let $\Lambda : L^2([0,\pi]) \longrightarrow L^2([0,\pi])$ be given by

$$(\Lambda g)(x) = \int_0^x g(y)\, dy.$$

Show that

$$(\Lambda f_n)(x) = \frac{x}{2} - \frac{\sin 2nx}{4n} \quad \text{and} \quad (\Lambda f)(x) = \frac{x}{2}$$

and

$\Lambda f_1, \Lambda f_2, \ldots$ converges uniformly (and strongly) to Λf.