§7.1

Let X be a Banach space and let

$F : X \to X$ be a Lipschitz continuous map.

Let $\bar{u} \in X$. The Cauchy problem is to find $u : \mathbb{R} \to X$ such that

$$\frac{d}{dt} u(t) = F(u(t)) \quad \text{and} \quad u(0) = \bar{u}.$$

Theorem 7.1: The Cauchy problem has a unique solution.

"As in the finite dimensional case, the global existence and uniqueness of a solution can be proved using the contraction mapping theorem".

Approximate solutions to the Cauchy problem

(a) Forward Euler approximations

Let $h \in \mathbb{R}_{>0}$ and $t_j = jh$ for $j \in \mathbb{Z}_{>0}$.

Let

$$x(t_{j+1}) = x(t_j) + h F(x(t_j)) \qquad \left(\begin{array}{l} \text{so } \dot{x}(t) = Fx(t_j) \\ \text{for } t \in [t_j, t_{j+1}] \end{array} \right)$$

(b) Backward Euler approximations

Let $h \in \mathbb{R}_{>0}$ and $t_j = jh$ for $j \in \mathbb{Z}_{>0}$.

Let

$$x(t_{j+1}) = x(t_j) + h\,F(x(t_{j+1})) \qquad \begin{pmatrix} \text{so } \dot{x}(t) = F(x(t_{j+1})) \\ \text{for } t \in [t_j, t_{j+1}] \end{pmatrix}$$

The backward Euler approximation is more work to compute but has better stability and convergence.

Example The case when $X = \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator. The solution to the Cauchy problem

$$\frac{d}{dt} u(t) = Au(t), \qquad u(0) = \bar{u}$$

is $u : \mathbb{R} \to \mathbb{R}^n$ given by

$$u(t) = e^{tA}\bar{u}, \quad \text{where } e^{tA} = 1 + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots.$$

Note: $A = \lim_{t \to 0^+} \frac{e^tA - I}{t}$, $e^{tA}e^{sA} = e^{(t+s)A}$, $e^{0A} = 1$.

If

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{then } e^{tA} = \begin{pmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{pmatrix},$$

$$\|A\| = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} \text{ and } \|e^{tA}\| = \max\{|e^{t\lambda_1}|, \ldots, |e^{t\lambda_n}|\}$$

<u>Example</u> Let $X = \ell^1$ and let $A : \ell^1 \to \ell^1$ be given by

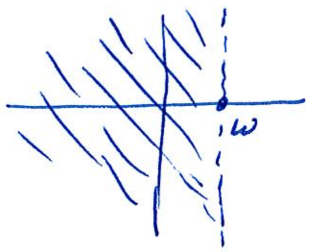
$$A(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots)$$

Then
$$e^{tA}(x_1, x_2, \ldots) = (e^{t\lambda_1} x_1, e^{t\lambda_2} x_2, \ldots)$$

Then

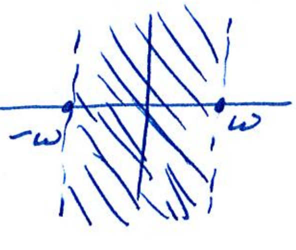
$$\|A\| = \sup\{|\lambda_1|, |\lambda_2|, \ldots\} \text{ and } \|e^{tA}\| = \sup\{|e^{t\lambda_1}|, |e^{t\lambda_2}|, \ldots\}.$$

(a) If there exists $w \in \mathbb{R}$ such that

λ_k is in  then

$$\|e^{tA}\| \le e^{tw} \quad \text{for } t \in \mathbb{R}_{\ge 0}$$

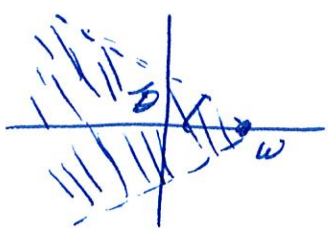
(b) If there exists $w \in \mathbb{R}_{\ge 0}$ such that

λ_k is in  then

$$\|e^{tA}\| \le e^{|t|w} \quad \text{for } t \in \mathbb{R},$$

since $|e^{t\lambda_k}| = |e^{t\,\mathrm{Re}(\lambda_k)}| \le e^{|t|w}$.

(c) If there exist $w \in \mathbb{R}$ and $\theta \in (0, \frac{\pi}{2})$ such that

λ_k is in the sector  then

$$\|e^{tA}\| \le e^{tw} \quad \text{for } t \in \mathbb{R}_{\ge 0}.$$

§7.2 and §7.4

A strongly continuous semigroup of linear operators on X is a homomorphism

$$\begin{aligned} S : \mathbb{R}_{\geq 0} &\to B(X,X) \\ t &\mapsto S_t \end{aligned} \quad \text{such that}$$

if $u \in X$ then $\begin{aligned} \mathbb{R}_{\geq 0} &\to X \\ t &\mapsto S_t u \end{aligned}$ is continuous.

The semigroup S is of type ω if S satisfies

if $t \in \mathbb{R}_{\geq 0}$ then $\|S_t\| \leq e^{t\omega}$

The semigroup S is contractive if S satisfies

if $t \in \mathbb{R}_{\geq 0}$ then $\|S_t\| \leq 1$.

The generator of S is $A : X \to X$ given by

$$Au = \lim_{t \to 0^+} \frac{S_t u - u}{t} \qquad (*)$$

Theorem 7.6 Let $\bar{u} \in X$, let S be a strongly continuous semigroup of linear operators on X and let $A : X \to X$ be given by $(*)$. Then

$\begin{aligned} u : \mathbb{R}_{\geq 0} &\to X \\ t &\mapsto S_t \bar{u} \end{aligned}$ is a solution to the Cauchy

problem $\frac{d}{dt} u(t) = Au(t), \quad u(0) = \bar{u}$.

Theorem 7.13 Let X be a Banach space and let $A : X \to X$ be a linear operator.

There exists a strongly continuous semigroup S of linear operators on X of type ω with generator A

if and only if

(a) A is defined on a dense subset of X

(b) $A : X \to X$ has closed graph.

From the proof: The semigroup S is constructed by

$$S_t u = \lim_{\lambda \to \infty} e^{tA_\lambda} u,$$

$$A_\lambda = -\lambda I + \lambda^2 R_\lambda = \lambda A R_\lambda, \quad \text{and}$$

$$R_\lambda = (\lambda I - A)^{-1}.$$

Theorem 7.14 Let S and $\hat{S}$ be strongly continuous semigroups of linear operators with the same generator A. Then $S = \hat{S}$.

For the proof of Theorem 7.13 do we need

$$\mathbb{R}_{>\omega} \subseteq \{\lambda \in \mathbb{R} \mid \lambda I - A \text{ is a bijection}\} ??$$

and if $\lambda > \omega$ then $\|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda - \omega}$??

§ 7.3

Let X be a Banach space and let

$A: X \to X$ be a linear operator.

The <u>backward Euler operator</u> for $h \in \mathbb{R}_{>0}$ is

$$E_h^- = (I - hA)^{-1}.$$

We expect and hope that if $\bar{u} \in X$ then

$$u(t) \approx (E_{t/n}^-)^{n} \bar{u} = (I - \tfrac{t}{n}A)^{-n} \bar{u} \quad \text{and}$$

$$u(t) = \cancel{\lim} \, S_t \bar{u} = \lim_{n \to \infty} (I - \tfrac{t}{n}A)^{-n} \bar{u}$$

is a solution to the Cauchy problem

$$\frac{d}{dt} u(t) = Au(t), \qquad u(0) = \bar{u}. \qquad (*).$$

Alternatively, let $h \in \mathbb{R}_{>0}$ and $\lambda = \frac{1}{h}$ and define

$$A_\lambda : X \to X \quad \text{by} \quad A_\lambda u = AE_h^- = A(I - hA)^{-1} u.$$

Then we hope that

$$u(t) = S_t \bar{u} = \lim_{\lambda \to \infty} e^{tA_\lambda} \bar{u}$$

is a solution to the Cauchy problem $(*)$

Here

$$e^{tA_\lambda} = 1 + tA_\lambda + \frac{t^2}{2!} A_\lambda^2 + \frac{t^3}{3!} A_\lambda^3 + \cdots$$

Let X be a Banach space and let
$A: X \to X$ be a linear operator.

The resolvent set of A is

$$\rho(A) = \{\lambda \in \mathbb{R} \mid \lambda I - A \text{ is a bijection}\}.$$

The resolvent operators are $R_\lambda : X \to X$ given by

$$R_\lambda u = (\lambda I - A)^{-1} u, \qquad \text{for } \lambda \in \rho(A)$$

Note (a) $A R_\lambda u = R_\lambda A u$ for $u \in Dom(A)$

(b) $\lambda R_\lambda = E^-_{1/\lambda}$

(c) $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$

(d) $R_\lambda R_\mu = R_\mu R_\lambda$

Theorem 7.11 Let S be a strongly continuous semigroup of linear operators on X of type ω and let A be the generator of S.

If $\lambda \in \mathbb{R}_{>\omega}$ then $\lambda \in \rho(A)$,

$$R_\lambda u = \int_0^\infty e^{-t\lambda} S_t u \, dt \quad \text{and} \quad \|R_\lambda\| \le \frac{1}{\lambda - \omega}.$$

Note: If $h < \frac{1}{\omega}$ then

$$E_h^- u = (I - hA)^{-1} u = \int_0^\infty \frac{e^{-t/h}}{h} S_t u \, dt$$