Lecture notes for functional analysis
Sem. II 2014

Bressan Chapter 1

§1.1 Think of

known vector

$$Ax = b$$

matrix

unknown vector

as an analogue of

known function $f : \Omega \to \mathbb{R}$

$$Lu = f$$

linear partial differential operator

unknown function

(I) Positivity:

If A is strictly positive definite then A is invertible and $Ax = b$ has a unique solution.

If L is strictly positive definite then $Lu = f$ has a unique solution $u \in H_0^1(\Omega)$ (see Ch. 8).

Note: L is elliptic $\Rightarrow$ L is strictly positive definite

(II) Fredholm alternative:

$Ax = b$ has a unique solution $\iff$ $Ax = 0$ has a unique solution. (which will be $x = 0$)

$\iff \ker A = 0$

Assume λ is Fredholm. Then

$$\lambda \text{ is injective} \iff \lambda \text{ is surjective.}$$

This is the main tool for
If L is a linear elliptic operation
$Lu = f$ has a unique solution $u \in H_0^1(\Omega)$. $\iff$ $Lu = 0$ has a unique solution.

(III) Diagonalization.

If A is diagonal with eigenvectors $v_1, \dots, v_n$
then
$$x = \sum_{k=1}^{n} \frac{1}{\lambda_k} \langle b, v_k \rangle v_k$$

where $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues.

If L is elliptic then

$$u = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle f, \varphi_k \rangle_{L^2} \varphi_k$$

§1.2

§1.3

$$\mathcal{C}^k = \left\{ \begin{array}{l} \text{bounded continuous functions with} \\ \text{bounded continuous partial derivatives} \\ \text{up to order } k \end{array} \right\}$$

is a vector space with norm

$$\|f\|_{\mathcal{C}^k} = \max_{\alpha_1+\cdots+\alpha_n \le k} \sup_{x\in\mathbb{R}^n} |\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} f(x)|.$$

Sometimes

Sobolev spaces $W^{k,p} = \left\{ \begin{array}{l} \text{functions } \cancel{\text{with}} \text{ whose derivatives} \\ \text{up to order } k \text{ lie in } \mathbb{L}^p \end{array} \right\}$

with norm

$$\|f\|_{W^{k,p}} = \left(\sum_{\alpha_1+\cdots+\alpha_n \le k} \int_{\mathbb{R}^n} |\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} f(x)|^p dx \right)^{1/p}$$

are better (see Chapter 8).

§1.4

(a) Construct a sequence of approximate solutions (u_n)

(b) Extract a convergent subsequence

(c) Show that the limit of the subsequence is a solution.

To accomplish (b) use

(1) Compactness, or

(2) Weak convergence and Banach-Alaoglu Theorem (Chapter 2) or

(3) weak norm versus strong norm
and Ascoli's theorem (Chapter 3) or
Rellich-Kondrachov compact embedding theorem (Chapter 8).

§1.2

The Cauchy problem

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \quad \text{with} \quad \vec{x}(0) = \vec{b}$$

has unique solution

$$\vec{x}(t) = e^{tA}\vec{b}, \quad \text{where} \quad e^{tA} = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{t^k A^k}{k!}$$

so that

$$e^{0A} = 1 \quad \text{and} \quad e^{tA}e^{sA} = e^{(t+s)A}.$$

If A has an orthonormal basis of eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ then

$$\vec{x}(t) = e^{tA}\vec{b} = \sum_{k=1}^{n} e^{t\lambda_k}\langle b, v_k\rangle v_k.$$

We have

$$e^{tA} = \lim_{n\to\infty}\left(1 - \frac{t}{n}A\right)^{-n} \quad \text{and}$$

$$e^{tA} = \lim_{\lambda\to\infty} e^{tA_\lambda} \quad \text{where} \quad A_\lambda = A\left(1 - \frac{1}{\lambda}A\right)^{-1}.$$

Parabolic evolution equations

$$\frac{d}{dt} u(t) = -L u(t) \quad \text{with} \quad u(0) = g$$

where $g \in L^2(\Omega)$ and $u = 0$ on $\partial\Omega$.

If L is elliptic (symmetric) then there is an orthonormal basis $\varphi_1, \varphi_2, \ldots$ of $L^2(\Omega)$ and

$$u(t) = S_t g = \sum_{k=1}^{\infty} e^{-t\lambda_k} \langle g, \varphi_k \rangle_{L^2} \varphi_k \quad \text{for } t \in \mathbb{R}_{\geq 0}$$

and

$$S_0 = 1 \quad \text{and} \quad S_t \circ S_s = S_{t+s} \quad \text{for } s, t \in \mathbb{R}_{\geq 0}.$$

Second order scalar ODE Cauchyish problem

$$\frac{d^2}{dt^2} \vec{x}(t) + A \vec{x}(t) = 0 \quad \text{with} \quad x(0) = \vec{a}, \quad \frac{d}{dt} \vec{x}(0) = \vec{b}$$

is helpfully rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}$$

and if A has orthonormal basis of eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ then

$$\vec{x}(t) = \sum_{k=1}^{n} c_k(t) v_k$$

where the $c_k(t)$ are computed by solving

$$\frac{d^2}{dt^2} c_k(t) + \lambda_k c_k(t) = 0 \quad \text{with} \quad c_k(0) = \langle \vec{a}, v_k \rangle \quad \text{and} \quad \frac{d}{dt} c_k(0) = \langle \vec{b}, v_k \rangle.$$

Hyperbolic initial value problem

$$u_{tt} + Lu = 0 \quad \text{with} \quad \begin{array}{l} u(0) = g, \\ u_t(0) = h, \end{array} \quad \text{and } u = 0 \text{ on } \partial\Omega.$$

If the elliptic operator L is symmetric,

and $\{\varphi_1, \varphi_2, \dots\}$ is an orthonormal basis of $L^2(\Omega)$

then

$$u(t) = \sum_{k=1}^{\infty} c_k(t)\,\varphi_k, \qquad \text{for } t \in \mathbb{R}_{\geq 0},$$

with $c_k(t)$ determined by

$$\frac{d^2}{dt^2} c_k(t) + \lambda_k c_k(t) = 0, \quad c_k(0) = \langle g, \varphi_k \rangle_{L^2}, \quad \frac{d}{dt} c_k(0) = \langle h, \varphi_k \rangle_{L^2}.$$