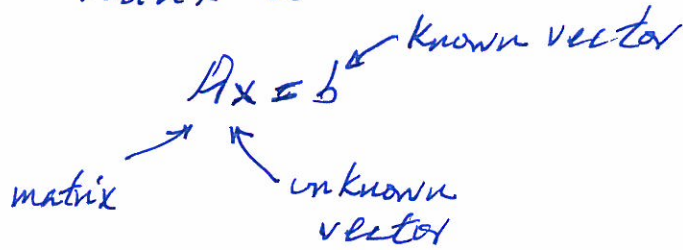


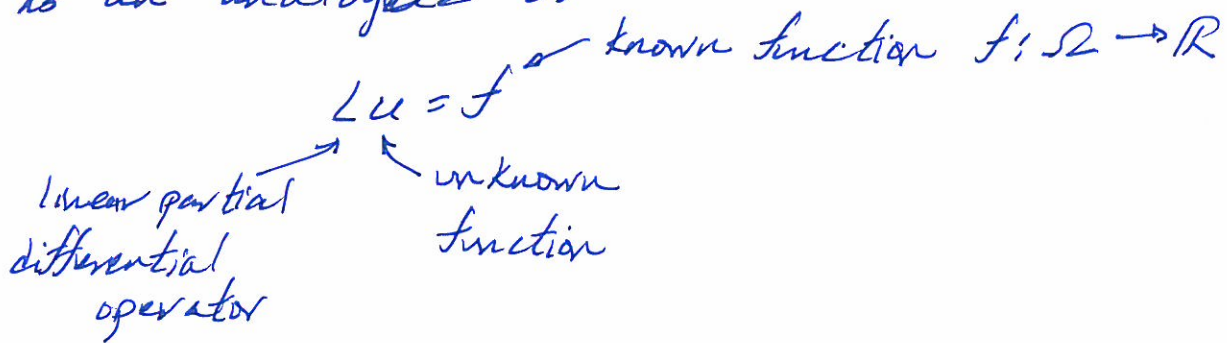
Lecture notes for functional analysis
Bressan Chapter 1 Sem. II 2014

①

§1.1 Think of



as an analogue of



(I) Positivity:

If A is strictly positive definite then A is invertible and $Ax = b$ has a unique solution.

If L is strictly positive definite then $Lu = f$ has a unique solution $u \in H_0^1(\Omega)$ (see Ch. 8).

Note: L is elliptic \Rightarrow L is strictly positive definite

(II) Fredholm alternative:

$Ax = b$ has a unique solution $\iff Ax = 0$ has a unique solution. (which will be $x = 0$)
 $\iff \ker A = 0$

(2)

Assume λ is Fredholm. Then

λ is injective $\Leftrightarrow \lambda$ is surjective.

This is the main tool for

If L is a linear elliptic operator

$Lu = f$ has a
unique solution

$u \in H_0^1(\Omega)$.

$\Leftrightarrow Lu = 0$ has a unique
solution.

(III) Diagonalization.

If A is diagonal with eigenvectors v_1, \dots, v_n

then $x = \sum_{k=1}^n \frac{1}{\lambda_k} \langle b, v_k \rangle v_k$

where $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues.

If L is elliptic then

$$u = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle f, \varphi_k \rangle_{L^2} \varphi_k$$

§ 1.2

§1.3

$$C^k = \left\{ \begin{array}{l} \text{bounded continuous functions with} \\ \text{bounded continuous partial derivatives} \\ \text{up to order } k \end{array} \right\}$$

is a vector space with norm

$$\|f\|_{C^k} = \max_{k_1 + \dots + k_n \leq k} \sup_{x \in \mathbb{R}^n} |\partial_{x_1}^{k_1} \dots \partial_{x_n}^{k_n} f(x)|.$$

Sometimes

Sobolev spaces $W^{k,p} \cong \left\{ \begin{array}{l} \text{functions with whose derivatives} \\ \text{up to order } k \text{ lie in } L^p \end{array} \right\}$

with norm

$$\|f\|_{W^{k,p}} = \left(\sum_{k_1 + \dots + k_n \leq k} \int_{\mathbb{R}^n} |\partial_{x_1}^{k_1} \dots \partial_{x_n}^{k_n} f(x)|^p dx \right)^{1/p}$$

are better (see Chapter 8).

§1.4

- Construct a sequence of approximate solutions (u_n)
- Extract a convergent subsequence
- Show that the limit of the subsequence is a solution.

To accomplish (b) use

- Compactness, or
- Weak convergence and Banach-Alaoglu Theorem (Chapter 2) or

(3) weak norm versus strong norm

and Ascoli's theorem (Chapter 3) or

Rellich-Kondrakov compact embedding theorem (Chapter 8).

§1.2

The Cauchy problem

$$\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) \quad \text{with} \quad \vec{x}(0) = \vec{b}$$

has unique solution

$$\vec{x}(t) = e^{tA} \vec{b}, \quad \text{where} \quad e^{tA} = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{t^k A^k}{k!}$$

so that

$$e^{0A} = 1 \quad \text{and} \quad e^{tA} e^{sA} = e^{(t+s)A}$$

If A has an orthonormal basis of eigenvectors v_1, \dots, v_n with eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$\vec{x}(t) = e^{tA} \vec{b} = \sum_{k=1}^n e^{t\lambda_k} \langle \vec{b}, v_k \rangle v_k.$$

We have

$$e^{tA} = \lim_{n \rightarrow \infty} \left(1 - \frac{t}{n} A\right)^{-n} \quad \text{and}$$

$$e^{tA} = \lim_{\lambda \rightarrow \infty} e^{tA_\lambda} \quad \text{where} \quad A_\lambda = A \left(1 - \frac{t}{\lambda} A\right)^{-1}.$$

(5)

Parabolic evolution equations

$$\frac{d}{dt} u(t) = -L u(t) \quad \text{with} \quad u(0) = g$$

where $g \in L^2(\Omega)$ and $u = 0$ on $\partial\Omega$.

If L is elliptic (symmetric) then there is an orthonormal basis $\varphi_1, \varphi_2, \dots$ of $L^2(\Omega)$ and

$$u(t) = S_t g = \sum_{k=1}^{\infty} e^{-t\lambda_k} \langle g, \varphi_k \rangle_{L^2} \varphi_k \quad \text{for } t \in \mathbb{R}_{\geq 0}$$

and

$$S_0 = I \quad \text{and} \quad S_t \circ S_s = S_{t+s} \quad \text{for } s, t \in \mathbb{R}_{\geq 0}.$$

Second order scalar ODE Cauchy problem

$$\frac{d^2}{dt^2} \vec{x}(t) + A \vec{x}(t) = 0 \quad \text{with} \quad \vec{x}(0) = \vec{a}, \quad \frac{d}{dt} \vec{x}(0) = \vec{b}$$

is helpfully rewritten as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix}$$

and if A has orthonormal basis of eigenvectors v_1, \dots, v_n with eigenvalues $\lambda_1, \dots, \lambda_n$ then

$$\vec{x}(t) = \sum_{k=1}^n c_k(t) v_k$$

where the $c_k(t)$ are computed by solving

$$\frac{d^2}{dt^2} c_k(t) + \lambda_k c_k(t) = 0 \quad \text{with} \quad c_k(0) = \langle \vec{a}, v_k \rangle \quad \text{and} \quad \frac{d}{dt} c_k(0) = \langle \vec{b}, v_k \rangle.$$

⑥

Hyperbolic initial value problem

$$u_{tt} + Lu = 0 \quad \text{with} \quad u|_D = g, \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega,$$
$$u_t|_D = h,$$

If the elliptic operator L is symmetric,
and $\{f_1, f_2, \dots\}$ is an orthonormal basis of $L^2(\Omega)$

then

$$u(t) = \sum_{k=1}^{\infty} c_k(t) f_k, \quad \text{for } t \in \mathbb{R}_{\geq 0},$$

with $c_k(t)$ determined by

$$\frac{d^2}{dt^2} c_k(t) + \lambda_k c_k(t) = 0, \quad c_k(0) = \langle g, f_k \rangle_{L^2}, \quad \frac{d}{dt} c_k(0) = \langle h, f_k \rangle_{L^2}.$$