Metric and Hilbert spaces

Assignment 2

Due Wednesday Oct 15 at 10am

1. Let

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \qquad \text{and}$$
$$B = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 1\}.$$

Determine, with proof, whether $X = A \cup B$, $Y = \overline{A} \cup \overline{B}$ and $Z = \overline{A} \cup B$ are connected subsets of $\mathbb{R}^2$ with the usual topology.

2. Let X and Y be topological spaces and assume that Y is Hausdorff. Let $f\colon X \to Y$ and $g\colon X \to Y$ be continuous functions.

(a) Show that the set $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

(b) Show that if $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous then

$$f - g \quad \text{is continuous.}$$

(c) Show that if $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous then

$$\{x \in X \mid f(x) < g(x)\} \quad \text{is open.}$$

3. Let X be a complete normed vector space over $\mathbb{R}$. A **sphere** in X is a set

$$S(a,r) = \{x \in X : d(x,a) = \|x - a\| = r\}, \qquad \text{for } a \in X \text{ and } r \in \mathbb{R}_{>0}.$$

(a) Show that each sphere in X is nowhere dense.

(b) Show that there is no sequence of spheres $\{S_n\}$ in X whose union is X.

(c) Give a geometric interpretation of the result in (b) when $X = \mathbb{R}^2$ with the Euclidean norm.

(d) Show that the result of (b) does not hold in every complete metric space X.

4. Prove that if X and Y are path connected then $X \times Y$ is also path connected.

5. Let $p \in \mathbb{R}_{>1}$ and define $q \in \mathbb{R}_{>1}$ by $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Define the normed vector space ℓ^p.

(b) Show that ℓ^p is a Banach space.

(c) Prove that the dual of ℓ^p is ℓ^q.

6. Let $X = C^1[0,1]$ and $Y = C[0,1]$ so that functions in X are continuously differentiable and functions in Y are continuous:

$$\begin{aligned} Y &= C[0,1], && \text{with norm given by} && \|f\| = \sup\{|f(t)| \mid t \in [0,1]\}, \text{ and} \\ X &= C^1[0,1], && \text{with norm given by} && \|f\|_0 = \|f\| + \|f'\|, \end{aligned}$$

where $f' = \frac{df}{dt}$. Let $D : X \to Y$ be the differentiation operator $Df = \frac{df}{dt}$.

(1) Show that $D : (X, \|\cdot\|_0) \to (Y, \|\cdot\|)$ is a bounded linear operator with $\|D\| = 1$.

(2) Show that $D : (X, \|\cdot\|) \to (Y, \|\cdot\|)$ is an unbounded linear operator. (*Hint:* Consider the sequence of elements t^n in X).

7. Let $\{a_1, a_2, \dots\}$ be a bounded sequence of complex numbers. Define an operator $T : l^2 \to l^2$ by;

$$T(b_1, b_2, \dots) = (0, a_1 b_1, a_2 b_2, \dots).$$

(1) Show that T is a bounded linear operator and find $\|T\|$.
(2) Compute the adjoint operator T^*.
(3) Show that if $T \neq 0$ then $T^*T \neq TT^*$.
(4) Find the eigenvalues of T^*.

8. Let $[a_{ij}]$ be an infinite complex matrix, $i, j = 1, 2, \dots,$ such that if $j \in \mathbb{Z}_{>0}$ then

$$c_j = \sum_i |a_{ij}| \quad \text{converges,} \qquad \text{and} \qquad c = \sup\{c_1, c_2, \dots\} < \infty.$$

Show that the operator $T\colon \ell^1 \to \ell^1$ defined by

$$T(b_1, b_2, \dots) = \Big(\sum_j a_{1j} b_j, \ \sum_j a_{2j} b_j, \ \dots \ \Big)$$

is a bounded linear operator and that $\|T\| = c$.