Tutorial Sheet 3

MAST30026 Metric and Hilbert Spaces
Semester II 2015
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(1) (The neighborhood filter is a filter) Let $(X, \mathcal{T})$ be a topological space and let $x \in X$.

(a) Define the neighborhood filter of x.

(b) Define filter.

(c) Show that the neighborhood filter of x is a filter on X.

(2) (Comparing the definitions of interior, closure, sup and inf) Let $(X, \mathcal{T}_X)$ be a topological space and let $E \subseteq X$. Let $(S, \leqslant)$ be a partially ordered set and let $F \subseteq S$.

(a) Define the interior of E.

(b) Define the closure of E.

(c) Define the sup, or least upper bound, of F.

(d) Define the inf, or greatest lower bound, of F.

State the definitions in (a-d) to be as similar to each other as possible.

(e) Give an example to show that $\sup(F)$ does not always exist.

(f) Give an example to show that $\inf(F)$ does not always exist.

(g) Show that the interior of E always exists.

(h) Show that the closure of E always exists.

(3) (closed is not the same as not open) Let $X = \mathbb{R}$, $Y = \mathbb{R}_{(0,1)} = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $Z = \mathbb{R}_{[0,1]} = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$ all with metric $d(x, y) = |x - y|$.

(a) Show that $(0, 1]$ is not open in X and not closed in X.

(b) Show that $(0, 1)$ is open in X and not closed in X.

(c) Show that $[0, 1]$ is closed in X and not open in X.

(d) Show that $\mathbb{R}$ is open in X and closed in X.

(e) Show that $(0,1)$ is closed in Y and not closed in X.

(f) Show that $[0,1]$ is open in Z and not open in X.

(g) Show that $\mathbb{R}$ is closed and open in $\mathbb{R}$.

(h) Show that $\mathbb{R}$ is closed and not open in $\mathbb{R}^2$.

(j) Show that the Cantor set is closed in $[0,1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}$.

(4) (closure of the open ball of radius 1 is not always distance $\leqslant 1$) Let (X,d) be a metric space. The *ball of radius* ϵ *centered at* x is

$$B_\epsilon(x) = \{y \in X \mid d(y,x) < \epsilon\}.$$

For a subset $A \subseteq X$ let $\overline{A}$ be the closure of A in X, in the metric space topology.

(a) Show that if $X = \mathbb{Z}$ with metric given by $d(x,y) = |x-y|$ then

$$\overline{B_1(0)} \neq \{y \in X \mid d(x,y) \leqslant 1\}.$$

(b) Show that if $X = \mathbb{R}$ with metric given by $d(x,y) = |x-y|$ then

$$\overline{B_1(0)} = \{y \in X \mid d(x,y) \leqslant 1\}.$$

(c) Let $X = \mathbb{R}^n$ with norm given by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x = (x_1, x_2, \ldots, x_n)$ and with metric given by $d(x,y) = \|x-y\|$ then

$$\overline{B_1(0)} = \{y \in X \mid d(x,y) \leqslant 1\}.$$

(5) (Interiors, closures and complements) Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$.

(a) Show that $\overline{E^c} = (E^\circ)^c$, by using the definition of closure.

(b) Show that $(E^c)^\circ = (\overline{E})^c$, by taking complements and using (a).

(6) (boundaries, dense sets and nowhere dense sets) Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.

The *boundary* of E is $\partial E = \overline{E} \cap \overline{E^c}$.

The set E is *dense in* X if $\overline{E} = X$.

The set E is *nowhere dense in* X if $(\overline{E})^\circ = \emptyset$.

Show that

(a) $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q}^\circ = \emptyset$.

(b) $(0,1]$ is dense in $[0,1]$.

(c) The boundary of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.
(d) The boundary of $(0,1]$ in $\mathbb{R}$ is $\{0,1\}$.
(e) $\mathbb{Z}_{>0}$ and $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$.
(f) $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.
(g) The Cantor set is nowhere dense in $[0,1]$.

(7) (dense implies not nowhere dense) Let $(X,\mathcal{T})$ be a topological space. Let $E \subseteq X$.

(a) Show that if E is dense in X then E is not nowhere dense in X.
(b) Show that if E is nowhere dense in X then E is not dense in X.
(c) Give an example of $E \subseteq X$ such that E is not dense in X and E is not nowhere dense in X.

(8) (intersection of two open dense sets is open and dense) Let (X,d) be a metric space and let $U \subseteq X$ and $V \subseteq X$. Show that if U and V are open and dense in X then $U \cap V$ is open and dense in X.

(9) (intersection of two dense sets is not necessarily dense) Let $X = \mathbb{R}$ with the usual metric and let $U = \mathbb{Q}$ and $V = \mathbb{Q}^c$. Show that U and V are dense in $\mathbb{Q}$ and $U \cap V = \emptyset$.

(10) (characterising bounded sets) Let (X,d) be a metric space.

(a) Define bounded subset of X.
(b) Show that a subset $E \subseteq X$ is bounded if and only if there exists an open ball $B_\epsilon(x)$ such that $E \subseteq B_\epsilon(x)$.

(11) (The subspace topology is a topology) Let $(X,\mathcal{T})$ be a topological space and let $Y \subseteq X$. Define the subspace topology on Y and show that it is a topology on Y.

(12) (The subspace metric is a metric) Let (X,d) be a topological space and let $Y \subseteq X$.

(a) Define the subspace metric on Y.
(b) Show that the subspace metric on Y is a metric on Y.
(c) Show that the metric space topology of Y with the subspace metric is the subspace topology on Y.

(13) (A subspace of a vector space) Let X be a $\mathbb{K}$-vector space. A *subspace* of X is a subset $V \subseteq X$ such that

(a) If $v_1, v_2 \in V$ then $v_1 + v_2 \in V$,

(b) If $v \in V$ and $c \in \mathbb{K}$ then $cv \in V$.

Show that V with the same operations of addition and scalar multiplication as in X is a vector space.

(14) (A subspace of a normed vector space is a normed vector space) Let X be a normed vector space. Let $V \subseteq X$ be a subspace. Show that V is a normed vector space with the same norm.

(15) (The product topology is a topology) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Define the product topology on $X \times Y$ and show that it is a topology on $X \times Y$.

(16) (The product metric is a metric) Let (X, d_X) and (Y, d_Y) be topological spaces.

(a) Define the product metric $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geqslant 0}$.

(b) Show that $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geqslant 0}$ is a metric on $X \times Y$.

(c) Let $\mathcal{T}_X$ be the metric space topology on (X, d_X) and let $\mathcal{T}_Y$ be the metric space topology on (X, d_Y). Show that the metric space topology of $(X \times Y, d)$ is the product topology on $X \times Y$.

(17) (The product topology on $\mathbb{R} \times \mathbb{R}$ is the standard topology on $\mathbb{R}^2$) Show that the product topology on $\mathbb{R} \times \mathbb{R}$ is equal to the standard topology on $\mathbb{R}^2$.

(18) (The product metric on $\mathbb{R} \times \mathbb{R}$ is not the standard metric on $\mathbb{R}^2$) Show that the product metric on $\mathbb{R} \times \mathbb{R}$ (where $\mathbb{R}$ has the standard metric) is *not* the standard metric on $\mathbb{R}^2$.

(19) (Metrics that produce the product topology) Let (X_1, d_1) and (X_2, d_2) be metric spaces. Let $Y = X_1 \times X_2$ and define

$$\begin{aligned}
d((x_1, x_2), (y_1, y_2)) &= d_1(x_1, y_1) + d_2(x_2, y_2), \\
\rho((x_1, x_2), (y_1, y_2)) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}, \\
\sigma((x_1, x_2), (y_1, y_2)) &= \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}
\end{aligned}$$

(a) Show that (Y, d), (Y, ρ) and (Y, σ) are metric spaces.

(b) Show that (Y, d), (Y, ρ) and (Y, σ) are the same as topological spaces.

(20) (Lipschitz equivalence implies topological equivalence) Let X be a set and let

$$d_1\colon X\times X\to\mathbb{R}_{\geqslant 0}\qquad\text{and}\qquad d_2\colon X\times X\to\mathbb{R}_{\geqslant 0}\qquad\text{be metrics on } X.$$

The metrics d_1 and d_2 are *topologically equivalent* if

the metric space topology on (X,d_1) and on (X,d_2) are the same.

The metrics d_1 and d_2 are *Lipschitz equivalent* if there exist $c_1,c_2\in\mathbb{R}_{>0}$ such that

$$\text{if } x,y\in X\quad\text{then}\quad c_1d_2(x,y)\leqslant d_1(x,y)\leqslant c_2d_1(x,y).$$

Show that if d_1 and d_2 are Lipschitz equivalent then d_1 and d_2 are topologically equivalent.

(21) (every metric space is topologically equivalent to a bounded metric space) A metric space (X,d) is *bounded* if it satisfies

there exists $M\in\mathbb{R}_{\geqslant 0}$ such that if $x_1,x_2\in X$ then $d(x_1,x_2)<M$.

Let (X,d) be a metric space and define $b\colon X\times X\to\mathbb{R}_{\geqslant 0}$ by

$$b(x,y)=\frac{d(x,y)}{1+d(x,y)}.$$

(a) Show that $b\colon X\times X\to\mathbb{R}_{\geqslant 0}$ is a metric on X.
(b) Show that the metric space topology of (X,b) and the metric space topology on (X,d) are the same.
(c) Show that (X,b) is a bounded metric space.

(22) (boundedness is not a topological property) A metric space (X,d) is *bounded* if it satisfies

there exists $M\in\mathbb{R}_{>0}$ such that if $x_1,x_2\in X$ then $d(x_1,x_2)<M$.

Let $X=\mathbb{R}$ and let $d\colon X\times X\to\mathbb{R}_{\geqslant 0}$ and $b\colon X\times X\to\mathbb{R}_{\geqslant 0}$ be the metrics on $\mathbb{R}$ given by

$$d(x,y)=|x-y|\qquad\text{and}\qquad b(x,y)=\frac{|x-y|}{1+|x-y|}.$$

Show that (X,d) and (X,b) have the same topology, that (X,d) is unbounded, and (X,b) is bounded.

(23) ($B(V,W)$ is a normed vector space) Let V and W be normed vector spaces. Show that

$$B(V,W)=\{\text{linear transformations } T\colon V\to W\mid \|T\|<\infty\}\qquad\text{where}$$

$$\|T\|=\sup\left\{\frac{\|Tv\|}{\|v\|}\mid v\in V\right\},$$

is a normed vector space.