

Examples

(a) The boundary of  $(0,1]$  in  $\mathbb{R}$  is  $\{0,1\}$



$$\overline{(0,1]} = [0,1] \quad ((0,1] \text{ is dense in } [0,1])$$

$$(0,1]^c = (-\infty, 0] \cup (1, \infty)$$

$$\overline{(0,1]^c} = (-\infty, 0] \cup [1, \infty) \quad \text{and} \quad \partial((0,1]) = \{0,1\}.$$

(b) If  $E = \mathbb{Q}$  in  $\mathbb{R}$  then  $\overline{\mathbb{Q}} = \mathbb{R}$ ,  $\mathbb{Q}^c = \mathbb{R}$  and

$$\partial \mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R}. \quad \text{Since } \overline{\mathbb{Q}} = \mathbb{R}, \mathbb{Q} \text{ is dense in } \mathbb{R}.$$

Also  $\mathbb{Q}^\circ = \emptyset$ , since  $\mathbb{Q}$  has no interior points.

(c) In  $\mathbb{R}$ ,

$$\overline{\mathbb{Z}} = \mathbb{Z} \quad \text{and} \quad \mathbb{Z}^\circ = \emptyset \quad \text{so} \quad (\overline{\mathbb{Z}})^\circ = \emptyset$$

So  $\mathbb{Z}$  is nowhere dense in  $\mathbb{R}$ .

(d) In  $\mathbb{R}^2$ ,

$$\overline{\mathbb{R}} = \mathbb{R} \quad \text{and} \quad \mathbb{R}^\circ = \emptyset \quad \text{since } \mathbb{R} \text{ has no interior points}$$

So  $\mathbb{R}$  is nowhere dense in  $\mathbb{R}^2$ .

The Cantor set

Let  $A = [0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$  and remove the middle third of  $A$  to get

$$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$



Now remove the middle third of each of the 2 components of  $A_1$  to get

$$A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$



The Cantor set is obtained by continuing this process:

$$C = \left( [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \cup [\frac{1}{9}, \frac{2}{9}] \cup [\frac{7}{9}, \frac{8}{9}] \cup [\frac{1}{27}, \frac{2}{27}] \cup [\frac{7}{27}, \frac{8}{27}] \cup \dots \right)^c$$

where the complement is taken  $[0, 1]$ .

HW: The set  $C$  is nowhere dense

and  $\text{Card}(C) = \text{Card}(\mathbb{R})$ .

Boundedness (only for metric spaces). (3)

A metric space  $(X, d)$  is bounded if it satisfies there exists  $M \in \mathbb{R}_{>0}$  such that if  $x_1, x_2 \in X$  then  $d(x_1, x_2) < M$ .

Example Let  $X = \mathbb{R}$  with the discrete metric

$$f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \text{ given by } d(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 \neq x_2 \\ 0, & \text{if } x_1 = x_2. \end{cases}$$

HW: Show that the metric space topology of  $(\mathbb{R}, f)$

is  $\mathcal{T}_f = \{ \text{all subsets of } \mathbb{R} \}$  (the discrete topology)

Example Let  $X = \mathbb{R}$  with the metric

$$b(x_1, x_2) = \frac{|x_1 - x_2|}{1 + |x_1 - x_2|}$$

To show:  $(X, b)$  is bounded.

To show: There exists  $M \in \mathbb{R}_{>0}$  such that

if  $x_1, x_2 \in X$  then  $b(x_1, x_2) < M$ .

Let  $M = 2$ .

To show: If  $x_1, x_2 \in X$  then  $b(x_1, x_2) < M$ .

Assume  $x_1, x_2 \in X$ .

To show:  $b(x_1, x_2) < M$ .

$$d(x_1, x_2) = \frac{|x_1 - x_2|}{1 + |x_1 - x_2|} = \frac{1 + |x_1 - x_2| - 1}{1 + |x_1 - x_2|} = 1 - \frac{1}{1 + |x_1 - x_2|}$$

$$< 1 < 2 = M.$$

So  $d(x_1, x_2) < M$ .

So  $(X, d)$  is bounded.

HW: Show that the metric space topology of  $(\mathbb{R}, d)$  is the same as the standard topology on  $\mathbb{R}$ ,

$\mathcal{J} = \{\text{unions of open intervals}\}$ .