Then $(-a)^2 > 0$, by Case 1.
So $a^2 > 0$, by Proposition 11.6.2 (f).

(c) To show: $1 \geqslant 0$.
$1 = 1^2 \geqslant 0$, by part (b).

(d) Assume $a \in \mathbb{F}$ and $a > 0$.
By part (b), $a^{-2} = (a^{-1})^2 > 0$.
So $a(a^{-1})^2 > a \cdot 0$, by (OFb).
So $a^{-1} > 0$, by (Fh) and Proposition 11.6.2 (a).

(e) Assume $a, b \in \mathbb{F}$ and $a \geqslant 0$ and $b \geqslant 0$.

$$\begin{aligned} a + b &\geqslant 0 + b, \quad \text{by (OFa)}, \\ &\geqslant 0 + 0, \quad \text{by (OFa)}, \\ &= 0, \quad \text{by (Fc)}. \end{aligned}$$

(f) Assume $a, b \in \mathbb{F}$ and $0 < a < b$.
So $a > 0$ and $b > 0$.
Then, by part (d), $a^{-1} > 0$ and $b^{-1} > 0$.
Thus, by (OFb), $a^{-1}b^{-1} > 0$.
Since $a < b$, then $b - a > 0$, by (OFa).
So, by (OFb), $a^{-1}b^{-1}(b-a) > 0$.
So, by (Fh), $a^{-1} - b^{-1} > 0$.
So, by (OFa), $a^{-1} > y^{-1}$.

□

1.4.5. If W is complete then $B(V,W)$ is complete. —

Theorem 1.4.5. — *Let $(V, \| \|)$ and $(W, \| \|)$ be normed vector spaces and let*

$$B(V,W) = \{\textit{linear transformations } T\colon V \to W \mid \|T\| < \infty\} \qquad \textit{where}$$

$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \right\}.$$

If W is complete then $B(V,W)$ is complete.

Proof. —

To show: If W is complete then $B(V,W)$ is complete.
Assume W is complete.
To show: If $T_1, T_2, \ldots$ is a Cauchy sequence in $B(V,W)$ then $T_1, T_2, \ldots$ converges.
Assume $T_1\colon V \to W$, $T_2\colon V \to W$, ... is a Cauchy sequence in $B(V,W)$.
To show: There exists $T\colon V \to W$ with $T \in B(V,W)$ such that $\lim_{n\to\infty} T_n = T$.
Define $T\colon V \to W$ by

$$T(x) = \lim_{n\to\infty} T_n(x).$$

To show: (a) If $x \in V$ then $T(x)$ exists.
(b) $T \in B(V,W)$.
(c) $\lim\limits_{n\to\infty} T_n = T$.

(a) Assume $x \in V$.
To show: $\lim_{n\to\infty} T_n(x)$ exists.
To show: $T_1(x), T_2(x), \ldots$ converges in W.

Since W is complete,

to show: $T_1(x), T_2(x), \ldots$ is Cauchy.

To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geqslant N}$ then $\|T_r(x) - T_s(x)\| < \epsilon$.

Assume $\epsilon \in \mathbb{R}_{>0}$.

Using that $T_1, T_2, \ldots$ is Cauchy, let N be such that

if $r, s \in \mathbb{Z}_{\geqslant N}$ then $\|T_r - T_s\| < \frac{\epsilon}{\|x\|}$.

To show: If $r, s \in \mathbb{Z}_{\geqslant N}$ then $\|T_r(x) - T_s(x)\| < \epsilon$.

Assume $r, s \in \mathbb{Z}_{\geqslant N}$.

To show: $\|T_r(x) - T_s(x)\| < \epsilon$.

$$\|T_r(x) - T_s(x)\| \leqslant \|T_r - T_s\| \cdot \|x\| < \frac{\epsilon}{\|x\|} \cdot \|x\| = \epsilon.$$

So $T_1(x), T_2(x), \ldots$ is Cauchy and, since W is complete, $T_1(x), T_2(x), \ldots$ converges.

So $T(x) = \lim_{n \to \infty} T_n(x)$ exists.

(b) To show: $T \in B(V, W)$.

To show: (ba) T is a linear transformation.

(bb) $\|T\| < \infty$.

(ba) To show: (baa) If $x_1, x_2 \in V$ then $T(x_1 + x_2) = T(x_1) + T(x_2)$.

(bab) If $c \in \mathbb{K}$ and $x \in V$ then $T(cx) = cT(x)$.

(baa) Assume $x_1, x_2 \in V$.

To show: $T(x_1 + x_2) = T(x_1) + T(x_2)$.

Since each T_n is a linear transformation and since

addition $\begin{array}{rccc} +\colon & W \times W & \to & W \\ & (w_1, w_2) & \mapsto & w_1 + w_2 \end{array}$ is continuous in W, then

$$\begin{aligned} T(x_1 + x_2) &= \lim_{n \to \infty} T_n(x_1 + x_2) = \lim_{n \to \infty} (T_n(x_1) + T_n(x_2)) \\ &= \lim_{n \to \infty} T_n(x_1) + \lim_{n \to \infty} T_n(x_2) = T(x_1) + T(x_2). \end{aligned}$$

(bab) Assume $c \in \mathbb{K}$ and $x \in V$.

To show: $T(cx) = cT(x)$.

Since each T_n is a linear transformation and since

scalar multiplication $\begin{array}{ccc} \mathbb{K} \times W & \to & W \\ (c, w) & \mapsto & cw \end{array}$ is continuous in W,

$$T(cx) = \lim_{n \to \infty} T_n(cx) = \lim_{n \to \infty} cT_n(x) = c \lim_{n \to \infty} T_n(x) = cT(x).$$

So T is a linear transformation.

(bb) To show: $\|T\| < \infty$.

To show: $\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} \mid x \in V \right\}$ exists in $\mathbb{R}_{\geqslant 0}$.

Since $\| \ \| \colon W \to \mathbb{R}_{\geqslant 0}$ is continuous,

$$\begin{aligned} \|Tx\| &= \|\lim_{n \to \infty} T_n(x)\| = \lim_{n \to \infty} \|T_n(x)\| \\ &\leqslant \lim_{n \to \infty} \|T_n\| \cdot \|x\| = \|x\| \big(\lim_{n \to \infty} \|T_n\| \big). \end{aligned}$$

By assumption, the sequence $T_1, T_2, \ldots$ is Cauchy and thus, since $\|T_r\| - \|T_s\| \leqslant \|T_r - T_s\|$, the sequence $\|T_1\|, \|T_2\|, \ldots$ is Cauchy.
Since $\mathbb{R}_{\geqslant 0}$ is complete, $\lim\limits_{n\to\infty} \|T_n\|$ exists.
So

$$\|T\| = \sup\left\{ \frac{\|Tx\|}{\|x\|} \mid x \in V \right\} \leqslant \lim_{n\to\infty} \|T_n\|,$$

and the right hand side exists in $\mathbb{R}_{\geqslant 0}$.
So $\|T\| < \infty$.
So $T \in B(V, W)$.

(c) To show: $\lim\limits_{n\to\infty} T_n = T$.
To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geqslant N}$ then $\|T - T_n\| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
Using that the sequence $T_1, T_2, \ldots$ is Cauchy,
let $N \in \mathbb{Z}_{>0}$ be such that if $m, n \in \mathbb{Z}_{\geqslant N}$ then $\|T_m - T_n\| < \frac{\epsilon}{2}$.
To show: If $n \in \mathbb{Z}_{\geqslant N}$ then $\|T - T_n\| < \epsilon$.
Assume $n \in \mathbb{Z}_{\geqslant N}$.
To show: $\|T - T_n\| < \epsilon$.
Since $\| \ \| \colon W \to \mathbb{R}_{\geqslant 0}$ is continuous,

$$\|T(x) - T_n(x)\| = \|\lim_{m\to\infty} (T_m(x) - T_n(x))\| = \lim_{m\to\infty} \|T_m(x) - T_n(x)\|.$$

Since $\|T_m - T_n\| < \frac{\epsilon}{2}$ for $m \in \mathbb{Z}_{\geqslant N}$ then $\|T_m(x) - T_n(x)\| < \frac{\epsilon}{2}\|x\|$ for $m \in \mathbb{Z}_{\geqslant N}$ and thus

$$\|T(x) - T_n(x)\| = \lim_{m\to\infty} \|T_m(x) - T_n(x)\| \leqslant \lim_{m\to\infty} \frac{\epsilon}{2}\|x\| = \frac{\epsilon}{2}\|x\|.$$

So $\|T - T_n\| \leqslant \frac{\epsilon}{2} < \epsilon$.
So $\lim\limits_{n\to\infty} \|T - T_n\| = 0$.
So $\lim\limits_{n\to\infty} T_n = T$.

□