The University of Melbourne

Department of Mathematics and Statistics

Semester Two 2017 SAMPLE EXAM 1

MAST30026 METRIC AND HILBERT SPACES

Examination duration:	Three hours
Reading time allowed:	Fifteen minutes
Number of pages:	3 (including this page)
Common Content:	No common content

Authorized Materials: No materials are authorized.

Instructions to Invigilators: One 14 page script book is to be given to each student initially. No written or printed material related to the subject may be brought into the examination. No mathematical tables or calculators may be used.

Instructions to Students:

- This paper has **6 questions**. You should attempt *all* problems.
- Aim for clear, concise and complete answers.
- Write all your solutions in the booklets provided.
- Number the questions clearly, and start each question on a new page.
- Use the *left* pages for rough working. Write material you wish to be marked on *right* pages only.

Question 1.

Let (X, d) be a metric space and let $A \subseteq X$.

10pts (a) Define the different kinds of compactness (including "closed" and "bounded").

20pts (b) Draw the diagram relating the different kinds of compactness.

20pts (c) Choose one of the implications in your diagram and prove it.

Question 2.

10pts (a) Carefully define a topology.

10pts (b) Carefully define a metric space.

10pts (c) Carefully define the ball of radius ϵ centred at x.

20pts (d) Let (X, d) be a metric space. Define

$$\mathcal{B} = \{B_\epsilon(x) \mid x \in X, \epsilon \in \mathbb{R}_{\geq 0}\}$$

and let

$$\mathcal{T} = \left\{ U \subseteq X \;\middle|\; \text{there exists } \mathcal{R} \subseteq \mathcal{B} \text{ such that } U = \bigcup_{B \in \mathcal{R}} B \right\}$$

Show that $\mathcal{T}$ is a topology on X.

Question 3.

20pts (a) Let $n \in \mathbb{Z}_{>0}$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is continuous.

20pts (b) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$ is continuous.

20pts (c) Let $f : S \to T$ be a function. Prove that the inverse function to f exists if and only if f is bijective.

Question 4.

10pts (a) Carefully define a uniformity.

10pts (b) Carefully define a metric space.

10pts (c) Carefully define the diagonal of width radius ϵ.

20pts (d) Let (X, d) be a metric space and let

$$\mathcal{X} = \{U \subseteq X \times X \mid \text{there exists } \epsilon \in \mathbb{R}_{>0}\ U \supseteq B_\epsilon\}$$

Show that $\mathcal{T}$ is a uniformity on X.

Question 5.

Assume that it is known that $\mathbb{R}_{\geq 0}$ is complete.

10pts (a) Prove that if $A \subseteq \mathbb{R}_{\geq 0}$ and $A \neq \emptyset$ and A is bounded then $\sup(A)$ exists.

10pts (b) Give an example (with proof) of an increasing sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geq 0}$ which does not converge.

10pts (c) Give an example (with proof) of a bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{R}_{\geq 0}$ which does not converge.

10pts (d) Prove that if $(a_1, a_2, \ldots)$ is an increasing and bounded sequence in $\mathbb{R}_{\geq 0}$ then $(a_1, a_2, a_3, \ldots)$ converges.

10pts (e) Give an example (with proof) of an increasing and bounded sequence $(a_1, a_2, \ldots)$ in $\mathbb{Q}_{\geq 0}$ which does not converge.

Question 6.

Let (X, d) be a metric space and let $(a_1, a_2, \ldots)$ be a sequence in X.

10pts (a) Carefully define cluster point and limit point of $(a_1, a_2, \ldots)$.

10pts (b) Prove that if z is a limit point of $(a_1, a_2, \ldots)$ then z is a cluster point of $(a_1, a_2, \ldots)$.

10pts (c) Carefully define Cauchy sequence and convergent sequence.

10pts (d) Prove that if $(a_1, a_2, \ldots)$ converges then $(a_1, a_2, \ldots)$ is Cauchy.

10pts (e) Carefully define complete metric space.

Question 7.

10pts (a) Carefully define a "topology on X" and a "uniformity on X".

10pts (b) Let (X, d) be a metric space. Carefully define the "metric space topology on X" and the "metric space uniformity on X".

10pts (c) Determine all the topologies on the set $X = \{0, 1\}$.

10pts (d) Determine all the uniformities on $X = \{0, 1\}$.

10pts (e) For each of the uniformities you gave in part (d), compute the uniform space topology.

Question 8.

10pts (a) Carefully define a normed vector space.

10pts (b) Carefully define a positive definite Hermitian inner product space.

10pts (c) Carefully state and prove the Cauchy-Schwarz inequality.

10pts (d) Carefully state and prove the Pythagorean theorem.

10pts (e) Carefully state and prove the parallelogram law.