(4a) Let X, Y, Z be topological spaces and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Show that $g \circ f: X \to Z$ is continuous.

Proof To show: If V is open in Z then $(g\circ f)^{-1}(V)$ is open in X.

Assume V is open in Z.

Since g is continuous then $g^{-1}(V)$ is open in Y.

Since f is continuous then $f^{-1}(g^{-1}(V))$ is open in X.

So

$$\begin{aligned} f^{-1}(g^{-1}(V)) &= \{x \in X \mid f(x) \in g^{-1}(V)\} \\ &= \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid (g\circ f)(x) \in V\} \\ &= (g\circ f)^{-1}(V) \text{ is open in } X. \end{aligned}$$

So $g\circ f$ is continuous. //

(4b) Let X, Y, Z be uniform spaces and let $f: X \to Y$ and $g: Y \to Z$ be uniformly continuous functions. Show that $g \circ f: X \to Z$ is uniformly continuous.

Proof To show: If V is an entourage for Z then $((g\circ f)\times(g\circ f))^{-1}(V)$ is an entourage for X.

Assume V is an entourage for Z.

Since g is uniformly continuous then
$(g\times g)^{-1}(V)$ is an entourage for Y.

Since f is uniformly continuous then
$(f\times f)^{-1}((g\times g)^{-1}(V))$ is an entourage for X.

So
$$
\begin{aligned}
(f\times f)^{-1}((g\times g)^{-1}(V)) &= \{(x_1,x_2)\in X\times X \mid (f(x_1), f(x_2))\in (g\times g)^{-1}(V)\} \\
&= \{(x_1,x_2)\in X\times X \mid (g(f(x_1)), g(f(x_2)))\in V\} \\
&= \{(x_1,x_2)\in X\times X \mid ((g\circ f)(x_1), (g\circ f)(x_2))\in V\} \\
&= \{(x_1,x_2)\in X\times X \mid ((g\circ f)\times(g\circ f))(x_1,x_2)\in V\} \\
&= ((g\circ f)\times(g\circ f))^{-1}(V)
\end{aligned}
$$
is an ~~ui~~entourage in X.

So $g\circ f$ is uniformly continuous. //

(4c) Let $X = \mathbb{R}$. Let $\mathcal{T}_1$ be the standard topology on X and let

$\mathcal{T}_2 = \{ U \subseteq \mathbb{R} \}$ be the discrete topology on $\mathbb{R}$ (all subsets of $\mathbb{R}$ are open for $\mathcal{T}_2$).

Let $\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$ be the identity map.

Then f is bijective and $f^{-1} = f$.

The map $\begin{aligned} f : (\mathbb{R}, \mathcal{T}_1) &\longrightarrow (\mathbb{R}, \mathcal{T}_2) \\ x &\longmapsto x \end{aligned}$

is not continuous since

$\{0\}$ is open for $\mathcal{T}_2$, but

$\{0\} = f^{-1}(\{0\})$ is not open for $\mathcal{T}_1$

since it is not a union of open ε-balls.