(6a) Assume $(f_1, f_2, \ldots)$ converges uniformly to f.

To show: $(f_1, f_2, \ldots)$ converges pointwise to f.

To show: If $x \in X$ then $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

Assume $x \in X$.

To show: $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

To show: $\lim\limits_{n\to\infty} \rho(f_n(x), f(x)) = 0$.

We know: If $\rho_\infty(g,h) = \sup\{\rho(g(y),h(y)) \mid y \in X\}$ then $\lim\limits_{n\to\infty} \rho_\infty(f_n, f) = 0$.

Since

$$\rho(f_n(x), f(x)) \le \sup\{\rho(f_n(y), f(y)) \mid y \in X\} = \rho_\infty(f_n, f)$$

then

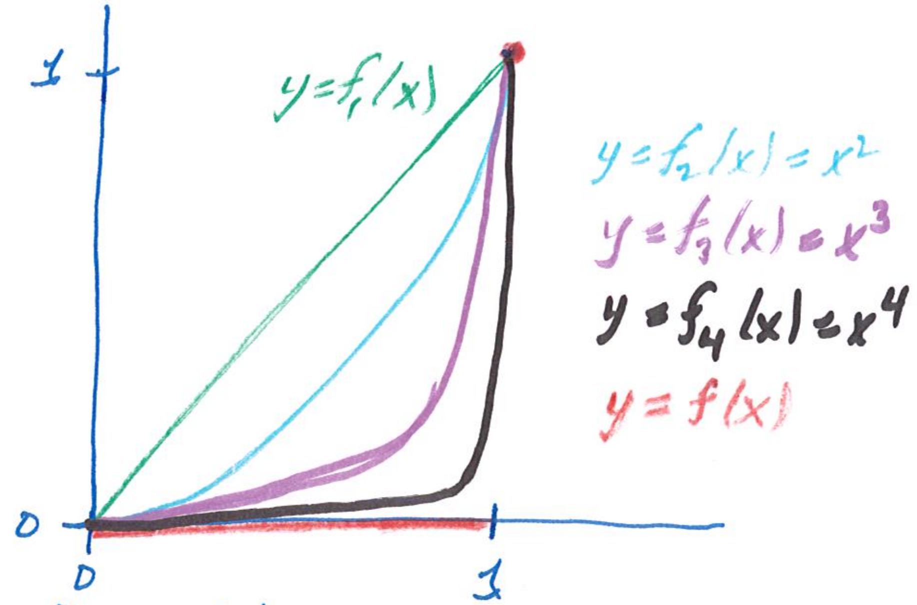
$$\lim_{n\to\infty} \rho(f_n(x), f(x)) \le \lim_{n\to\infty} \rho_\infty(f_n, f) = 0.$$

Since $\rho(f_n(x), f(x)) \in \mathbb{R}_{\ge 0}$ then $0 \le \lim\limits_{n\to\infty} \rho(f_n(x), f(x))$.

So $\lim\limits_{n\to\infty} \rho(f_n(x), f(x)) = 0$.

So $(f_1, f_2, \ldots)$ converges pointwise to f. //

(6b)



Let $x \in [0,1]$ with $x \neq 1$.

$$\lim_{n\to\infty} \rho(f_n(x), f(x)) = \lim_{n\to\infty} |x^n - 0|$$

Let $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $|x^n - 0| < \varepsilon$.

Let $N_1 \in \mathbb{Z}_{>0}$ be such that $|x| < 1 - \frac{1}{N_1+1}$

Let $N_2 \in \mathbb{Z}_{>0}$ be such that $\frac{1}{N_2} < \varepsilon$

Let $N \cancel{= \max(N_1, N_2)} = N_1 N_2$

To show: If $n \in \mathbb{Z}_{\geq N}$ then $|x^n| < \varepsilon$.

Assume $n \in \mathbb{Z}_{\geq N}$.

Then

$$|x^n| < \left(1 - \frac{1}{N_1+1}\right)^n = \left(\frac{N_1+1-1}{N_1+1}\right)^n = \left(\frac{N_1}{N_1+1}\right)^n$$

$$= \left(\frac{1}{1+\frac{1}{N_1}}\right)^n = \frac{1}{1+n\frac{1}{N_1}+\cdots+\left(\frac{1}{N_1}\right)^n}$$

$$< \frac{1}{1+\frac{n}{N_1}} = \frac{N_1}{n+N_1} \leq \frac{N_1}{N_1N_2+N_1} = \frac{1}{N_2+1} < \varepsilon.$$

So $\lim\limits_{n\to\infty} \rho(f_n(x), f(x)) = \lim\limits_{n\to\infty} |x^n - 0| = 0$

when $x \neq 1$.

If $x=1$ then

$\lim\limits_{n\to\infty} \rho(f_n(1), f(1)) = \lim\limits_{n\to\infty} |1^n - 1| = \lim\limits_{n\to\infty} 0 = 0.$

So $(f_1, f_2, \ldots)$ converges pointwise to f.

(6c) To show: $(f_1, f_2, \ldots)$ does not converge uniformly to f.

To show: $\lim\limits_{n\to\infty} \rho_\infty(f_n, f) \neq 0$.

To show: If $n \in \mathbb{Z}_{>0}$ then $\rho_\infty(f_n, f) = 1$.

Assume $n \in \mathbb{Z}_{>0}$.

To show: $\rho_\infty(f_n, f) = 1$.

To show: $\sup\{\rho(f_n(y), f(y)) \mid y \in [0,1]\} = 1$.

To show: $\sup\{ y^n - 0 \mid y \in [0,1)\} = 1$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $y \in [0,1)$ such that $y^n > 1-\varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $y \in [0,1)$ such that $y^n > 1-\varepsilon$.

Let $y \in [0,1)$ with $y > 1 - \frac{\varepsilon}{n-1}$, i.e. $y = 1 - \frac{\varepsilon}{2(n-1)}$.

To show: $y^n > 1-\varepsilon$.

To show: $1 - y^n < \varepsilon$.

$$1 - y^n = (1-y)(1+y+y^2+\cdots+y^{n-1})$$
$$< (1-y)(1+1+1+\cdots+1) = (1-y)(n-1)$$
$$= \left(1-\left(1-\frac{\varepsilon}{2(n-1)}\right)\right)(n-1) = \frac{\varepsilon}{2(n-1)}\cdot(n-1) = \frac{\varepsilon}{2} < \varepsilon$$

So $1-y^n > 1-\varepsilon$.

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So $\sup\{\rho(f_n(y), f(y)) \mid y \in [0,1]\} = 1$

So $\rho_\infty(f_n, f) = 1$

So $\lim\limits_{n\to\infty} \rho_\infty(f_n, f) = 1$.

So $\lim\limits_{n\to\infty} \rho_\infty(f_n, f) \neq 0$.

So $(f_1, f_2, \ldots)$ does not converge uniformly to f.