(1a) $\ell^p = \{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|x\|_p < \infty\}$

where
$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$$

The set ℓ^p is an $\mathbb{R}$-vector space with the operations

$x + y = (x_1 + y_1, x_2 + y_2, \ldots)$ for $x = (x_1, x_2, \ldots) \in \ell^p$ and $y = (y_1, y_2, \ldots) \in \ell^p$

and $cx = (cx_1, cx_2, \ldots)$ for $c \in \mathbb{R}$ and $x = (x_1, x_2, \ldots) \in \ell^p$.

There are many conditions to be checked (for example, $c(x+y) = cx + cy$) to verify that ℓ^p is an $\mathbb{R}$-vector space, but, for this question on this homework, in this context, it is much more important to check that $\| \ \|_p$ is a norm on ℓ^p.

(1b) To show: $\| \ \|_p$ is a norm on ℓ^p.

To show: (ba) If $x \in \ell^p$ and $\|x\|_p = 0$ then $x = 0$.

(bb) If $x \in \ell^p$ and $c \in \mathbb{R}$ then $\|cx\|_p = |c|\,\|x\|_p$.

(bc) If $x, y \in \ell^p$ then $\|x+y\|_p \le \|x\|_p + \|y\|_p$

(ba) Assume $x \in \ell^p$

To show: If $x \neq 0$ then $\|x\|_p \neq 0$.

Assume $x = (x_1, x_2, \ldots) \neq 0$.

Then there exists $k \in \mathbb{Z}_{>0}$ such that $x_k \neq 0$.

So $|x_k| > 0$

So $0 < |x_k|^p \le \sum_{i=1}^{\infty} |x_i|^p$.

So $\sum_{i=1}^{\infty} |x_i|^p \neq 0$.

So $\|x\|_p \neq 0$.

(bb) Assume $x \in \ell^p$ and $c \in \mathbb{R}$.

Write $x = (x_1, x_2, \ldots)$

To show: $\|cx\|_p = |c|\,\|x\|_p$

$$\|cx\|_p = \left(\sum_{i=1}^{\infty} |cx_i|^p\right)^{1/p} = \left(\sum_{i=1}^{\infty} |c|^p |x_i|^p\right)^{1/p}$$

$$= \left(|c|^p \cdot \left(\sum_{i=1}^{\infty} |x_i|^p\right)\right)^{1/p} = |c| \cdot \|x\|_p$$

where, for the third equality we have used that multiplication in $\mathbb{R}$ is continuous, since the infinite sums are limits.

(bc) Let $x = (x_1, x_2, \ldots) \in \ell^p$ and

$y = (y_1, y_2, \ldots) \in \ell^p$.

Let $\|x\|_{p,n} = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ and $\|y\|_{p,n} = \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$.

Then $(\|x\|_{p,1}, \|x\|_{p,2}, \ldots)$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ with limit $\|x\|_p$

and $(\|y\|_{p,1}, \|y\|_{p,2}, \ldots)$ is an increasing sequence in $\mathbb{R}_{\geq 0}$ with limit $\|y\|_p$.

By continuity of addition in $\mathbb{R}_{\geq 0}$

$$\lim_{n\to\infty} \|x\|_{p,n} + \lim_{n\to\infty} \|y\|_{p,n} = \|x\|_p + \|y\|_p.$$

By Theorem 1.17.1 part (b), which is proved in part III of the class notes, p. 26–28,

$$\|x+y\|_{p,n} \leq \|x\|_{p,n} + \|y\|_{p,n}$$

So $\|x+y\|_{p,n} \leq \|x\|_{p,n} + \|y\|_{p,n} \leq \|x\|_p + \|y\|_p$.

So $\|x+y\|_p = \lim\limits_{n\to\infty} \|x+y\|_{p,n} \leq \|x\|_p + \|y\|_p$.

(c) To show: ℓ^p is not an inner product space if $p \neq 2$.

Theorem 1.33.1 part (d) on p. 53, Part III of the notes:

If $(V, \langle\ \rangle)$ is an inner product space then if $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Let $x = (1, 0, \ldots)$ and $y = (0, 1, 0, \ldots)$ so that

$x+y = (1, 1, 0, \ldots)$ and $x-y = (1, -1, 0, \ldots)$

Then

$$\|x+y\|_p^2 + \|x-y\|_p^2 = ((1^p+1^p)^{1/p})^2 + ((1^p+1^p)^{1/p})^2$$

$$= 2^{2/p} + 2^{2/p} = 2 \cdot 2^{2/p}$$

and

$$2\|x\|_p^2 + 2\|y\|_p^2 = 2(1^p)^{1/p} + 2(1^p)^{1/p} = 4.$$

Now

$2 \cdot 2^{2/p} = 2 \cdot 2$ if and only if $\frac{2}{p} = 1$

if and only if $p = 2$.

So by Theorem 1.33.1 quoted above,

if $p \neq 2$ then ℓ^p is not an inner product space since it does not satisfy the parallelogram law.