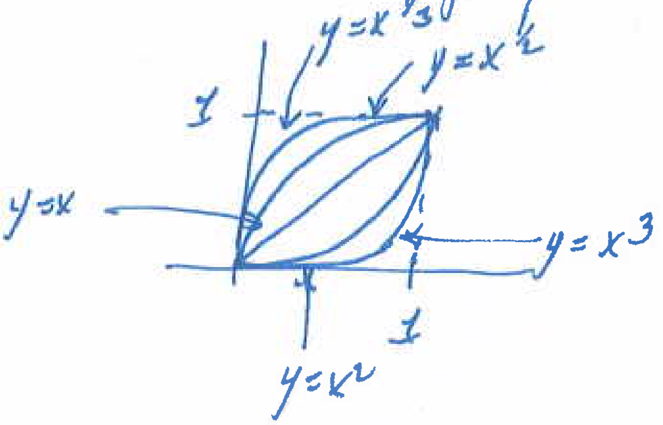
(2a) Let $p \in \mathbb{R}_{>0}$. The <u>p-norm</u> on $\mathbb{R}^2$ is

$\| \ \|_p : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ given by

$$\|(x_1, x_2)\|_p = \left(|x_1|^p + |x_2|^p\right)^{1/p}.$$

(2b) Use that the graphs of $y = x^p$ are



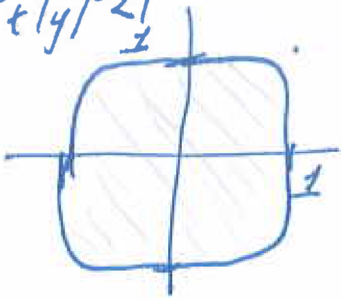
To determine the graphs of $y = 1 - x^p$ and $y^p = 1 - x^p$.

The ball of radius 1 is

$$\begin{aligned} B_1(0) &= \{(x,y) \in \mathbb{R}^2 \mid (|x|^p + |y|^p)^{1/p} \leq 1\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid |x|^p + |y|^p \leq 1\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid |y|^p \leq 1 - |x|^p\} \end{aligned}$$

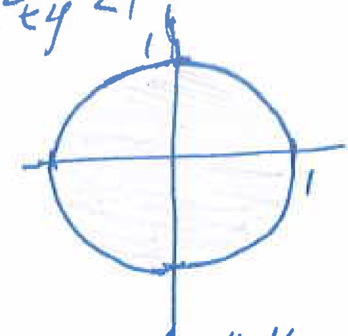
and we will use the symmetry of $|x|^p + |y|^p = 1$ about the x-axis $(x \mapsto -x)$ and about the y-axis $(y \mapsto -y)$ to conclude that the $B_1(0)$ for the ~~metric~~ norms $\| \ \|_1$, $\| \ \|_2$, $\| \ \|_3$, $\| \ \|_{\frac{1}{2}}$, $\| \ \|_{\frac{1}{3}}$ are as follows.

$|x|^3+|y|^3<1$



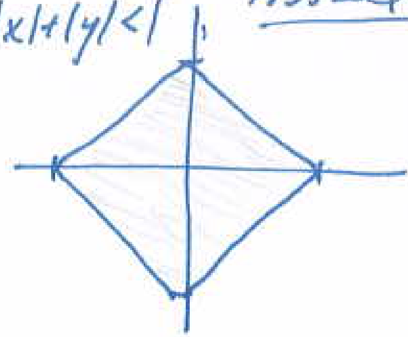
$B_1(0)$ for $\| \|_3$

$x^2+y^2<1$



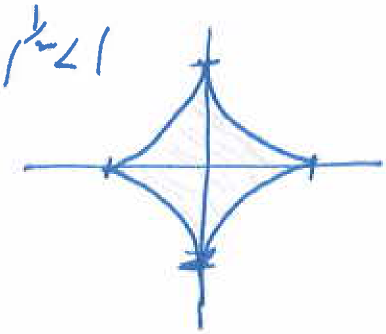
$B_1(0)$ for $\| \|_2$

$|x|+|y|<1$



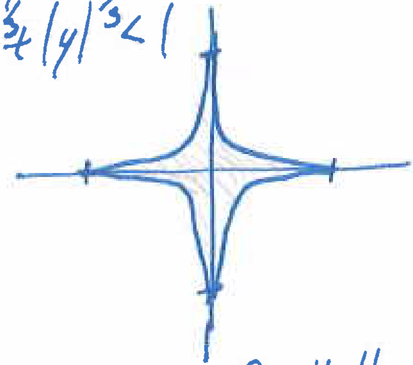
$B_1(0)$ for $\| \|_1$

$|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1$



$B_1(0)$ for $\| \|_{\frac{1}{2}}$

$|x|^{\frac{1}{3}}+|y|^{\frac{1}{3}}<1$

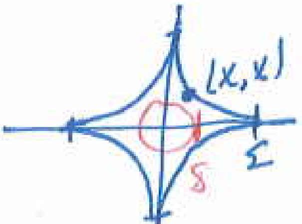


$B_1(0)$ for $\| \|_{\frac{1}{3}}$

(2c) To show that the topologies are equivalent the following pictures are helpful.

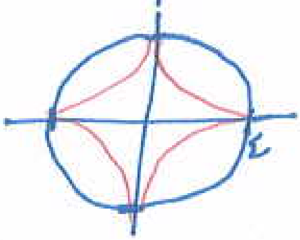
(A) Since

(x,x) with $(x^{\frac{1}{3}}+x^{\frac{1}{3}})^3=\varepsilon$



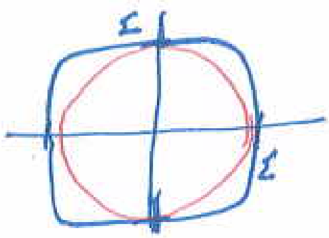
then $B_\delta^{\| \|_2}(0) \subseteq B_\varepsilon^{\| \|_{\frac{1}{3}}}(0)$

(B) Since



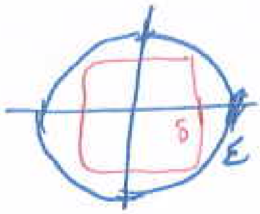
then $B_\varepsilon^{\| \|_{\frac{1}{3}}}(0) \subseteq B_\varepsilon^{\| \|_2}(0)$

(C) Since



then $B_\varepsilon^{\| \|_2}(0) \subseteq B_\varepsilon^{\| \|_3}(0)$

(D)

Since  then $B_\delta^{\|\ \|_3}(0) \subseteq B_\varepsilon^{\|\ \|_2}(0)$.

Let $\mathcal{T}_p$ be the topology generated by the open balls $B_\varepsilon^{\|\ \|_p}(x) = \{ y \in \mathbb{R}^2 \mid \|y-x\|_p < \varepsilon \}$ of radius ε centred at x for the norm $\|\ \|_p$.

Let $\mathcal{T}_2$ be the topology generated by the $B_\varepsilon^{\|\ \|_2}(x)$, the open balls for the norm $\|\ \|_2$.

To show: $\mathcal{T}_p = \mathcal{T}_2$

Case 1: $p > 2$

Case 2: $p < 2$.

Case 2: To show: (a) $\mathcal{T}_p \subseteq \mathcal{T}_2$

(b) $\mathcal{T}_2 \subseteq \mathcal{T}_p$.

(a) To show: If $U \in \mathcal{T}_p$ then $U \in \mathcal{T}_2$.

Assume $U \in \mathcal{T}_p$.

To show: If $x \in U$ then x is an interior point of U in the $\|\ \|_2$ ~~metric~~ norm.

Assume $x \in U$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that $B_\delta^{\|\ \|_2}(x) \subseteq U$.

We know there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon^{\|\ \|_p}(x) \subseteq \mathcal{U}$.

Let $\delta \in \mathbb{R}_{>0}$ such that $\delta^p + \delta^p < \varepsilon^p$, i.e.

$2\delta^p < \varepsilon^p$ so that $\delta < \frac{\varepsilon}{2^{1/p}}$.

See picture (A).

To show: $B_\delta^{\|\ \|_2}(x) \subseteq B_\varepsilon^{\|\ \|_p}(x)$

To show: If $p \in B_\delta^{\|\ \|_2}(x)$ then $p \in B_\varepsilon^{\|\ \|_p}(x)$.

Assume $p = (p_1, p_2) \in B_\delta^{\|\ \|_2}(x)$, with $x = (x_1, x_2)$.

Then $d_2(p,x) = (|p_1 - x_1|^2 + |p_2 - x_2|^2)^{1/2} < \delta$.

So $|p_1 - x_1|^2 + |p_2 - x_2|^2 < \delta^2$.

So $|p_1 - x_1|^2 < \delta^2$ and $|p_2 - x_2|^2 < \delta^2$.

To show: $p \in B_\varepsilon^{\|\ \|_p}(x)$.

To show: $(|p_1 - x_1|^p + |p_2 - x_2|^p)^{1/p} < \varepsilon$.

To show: $|p_1 - x_1|^p + |p_2 - x_2|^p < \varepsilon^p$.

$$|p_1 - x_1|^p + |p_2 - x_2|^p = (|p_1 - x_1|^2)^{p/2} + (|p_2 - x_2|^2)^{p/2}$$
$$< (\delta^2)^{p/2} + (\delta^2)^{p/2} = 2\delta^p < \varepsilon^p.$$

So $B_\delta^{\|\ \|_2}(x) \subseteq B_\varepsilon^{\|\ \|_p}(x) \subseteq \mathcal{U}$.

So $\|\varphi\| \leq \max\{|a|, |b|\}$

(b) To show: $\|\varphi\| \geq \max\{|a|, |b|\}$.

Since $|\varphi(1,0)| = |a| = |a| \, \|(1,0)\|_1$

then $\|\varphi\| \geq |a|$

Since $|\varphi(0,1)| = |b| = |b| \cdot \|(0,1)\|_1$

then $\|\varphi\| \geq |b|$.

So $\|\varphi\| \geq \max\{|a|, |b|\}$.

So $\|\varphi\| = \max\{|a|, |b|\}$.

(ii) $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\varphi(x_1, x_2) = ax_1 + bx_2$.

Assume $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.

To show: $\|\varphi\| = |a| + |b|$.

(a) Let $x = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\|\varphi(x)\| = |ax_1 + bx_2| \leq |a|\,|x_1| + |b|\,|x_2|$$
$$\leq (|a| + |b|) \max\{|x_1|, |x_2|\}$$
$$\leq (|a| + |b|) \cdot \|x\|_\infty .$$

So $\|\varphi\| \leq (|a| + |b|)$.

(b) To show: $\|\varphi\| \geq (|a| + |b|)$.

Let
$$x=(x_1,x_2)=\begin{cases}(1,1), & \text{if } a\in\mathbb{R}_{\geq 0} \text{ and } b\in\mathbb{R}_{\geq 0},\\ (1,-1), & \text{if } a\in\mathbb{R}_{\geq 0} \text{ and } b\in\mathbb{R}_{\leq 0},\\ (-1,1), & \text{if } a\in\mathbb{R}_{\leq 0} \text{ and } b\in\mathbb{R}_{\geq 0},\\ (-1,-1), & \text{if } a\in\mathbb{R}_{\leq 0} \text{ and } b\in\mathbb{R}_{\leq 0}.\end{cases}$$

Then $|x_1|=1$ and $|x_2|=1$ and

$$|\varphi(x)| = \big|\,|a|+|b|\,\big| = |a|+|b|$$
$$= (|a|+|b|)\cdot\max\{|x_1|, |x_2|\}$$
$$= (|a|+|b|)\,\|x\|_\infty.$$

So $\|\varphi\| \geq (|a|+|b|)$.

(iii) Assume $\|x\|_p = (|x_1|^p+|x_2|^p)^{1/p}$ and $\frac{1}{p}+\frac{1}{q}=1$.

Let $\varphi:\mathbb{R}^2\to\mathbb{R}$ be given by $\varphi(x_1,x_2)=ax_1+bx_2$.

To show: $\|\varphi\| = (|x_1|^q+|x_2|^q)^{1/q}$.

To show: (a) $\|\varphi\| \leq (|x_1|^q+|x_2|^q)^{1/q}$

(b) $\|\varphi\| \geq (|x_1|^q+|x_2|^q)^{1/q}$

(a) Let $x=(x_1,x_2)$. Then

$$|\varphi(x)| = |ax_1+bx_2| \leq (|a|^q+|b|^q)^{1/q}(|x_1|^p+|x_2|^p)^{1/p}$$

by the Hölder inequality proved as part (b) of Theorem 1.17.1 on pages 26-28 of Part III of the Notes.

So $|\varphi(x)| \leq (|a|^q + |b|^q)^{1/q} \, \|x\|_p$.

So $\|\varphi\| \leq (|a|^q + |b|^q)^{1/q}$

(b) To show: $\|\varphi\| \geq (|a|^q + |b|^q)^{1/q}$.

Let $x = (\pm|a|^{q/p}, \pm|b|^{q/p})$. Then

$$|\varphi(x)| = (\pm 1)a \cdot |a|^{q/p} + (b\,|b|^{q/p})(\pm 1)$$
$$= |a|^{1+q/p} + |b|^{1+q/p}$$
$$= |a|^{q(\frac{1}{q}+\frac{1}{p})} + |b|^{q(\frac{1}{q}+\frac{1}{p})}$$
$$= |a|^q + |b|^q$$
$$= (|a|^q + |b|^q)^{\frac{1}{q}+\frac{1}{p}}$$
$$= (|a|^q + |b|^q)^{\frac{1}{q}} \left((|a|^{q/p})^p + (|b|^{q/p})^p \right)^{1/p}$$
$$= (|a|^q + |b|^q)^{\frac{1}{q}} \left(|x_1|^p + |x_2|^p \right)^{1/p}$$
$$= (|a|^q + |b|^q)^{\frac{1}{q}} \|x\|_p.$$

(the signs $\pm$ are chosen so that $(\pm 1)a \cdot a^{q/p} = |a|^{1+q/p}$ and $b \cdot |b|^{q/p}(\pm 1) = |b|^{1+q/p}$.)

So $\|\varphi\| \geq (|a|^q + |b|^q)^{1/q}$.

So $\|\varphi\| = (|a|^q + |b|^q)^{1/q}$.