Metric and Hilbert Spaces: Lecture 11

Subspaces

Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.
The <u>subspace topology</u> on A is

$$\mathcal{T}_A = \{ U \cap A \mid U \in \mathcal{T} \}.$$

Let (X, d) be a metric space. Let $A \subseteq X$.
The <u>subspace metric</u> on A is

$d_A : A \times A \to \mathbb{R}_{\geq 0}$ given by $d_A(a_1, a_2) = d(a_1, a_2)$.

Product spaces

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

$$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}.$$

The <u>product topology on $X \times Y$</u> has

$$\mathcal{N}((x,y)) = \left\{ N \subseteq X \times Y \;\middle|\; \begin{array}{l} \text{there exists } N_x \in \mathcal{N}(x) \\ \text{and } N_y \in \mathcal{N}(y) \text{ such that} \\ N_x \times N_y \subseteq N \end{array} \right\}$$

and

$$\mathcal{T}_{X \times Y} = \left\{ U \subseteq X \times Y \;\middle|\; \begin{array}{l} \text{if } (x,y) \in U \text{ then there exists} \\ N \in \mathcal{N}(x,y) \text{ with } N \subseteq U \end{array} \right\}$$

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The set of <u>open rectangles</u> in $X \times Y$ is

$$\mathcal{B} = \{ U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \}.$$

Then

$$\mathcal{T}_{X \times Y} = \left\{ Z \subseteq X \times Y \;\middle|\; \begin{array}{l} \text{there exists } \mathcal{S} \subseteq \mathcal{B} \\ \text{such that } Z = \left(\bigcup_{R \in \mathcal{S}} R \right) \end{array} \right\}$$

In English: Z is open in $X \times Y$ if Z is a union of open rectangles.

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let

$$\mathcal{T}_{X \times Y} = \left\{ Z \subseteq X \times Y \;\middle|\; \begin{array}{l} \text{there exists } \mathcal{S} \subseteq \mathcal{B} \text{ such that} \\ Z = \left(\bigcup_{R \in \mathcal{S}} R \right) \end{array} \right\}$$

Then $\mathcal{T}_{X \times Y}$ is a topology on $X \times Y$.

<u>Products for metric spaces</u>

Let (X, d_X) and (Y, d_Y) be metric spaces.

Define

$$d_1 : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0} \quad \text{and}$$

$$d_2 : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0} \quad \text{by}$$

$$d_1((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

$$d_2((x_1,y_1),(x_2,y_2)) = \left(d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2\right)^{\frac{1}{2}}.$$

<u>Proposition</u> (a) d_1 and d_2 are both metrics on $X \times Y$.

(b) Let $\mathcal{T}_1$ be the metric space topology for $(X \times Y, d_1)$

Let $\mathcal{T}_2$ be the metric space topology for $(X \times Y, d_2)$

Then $\mathcal{T}_1 = \mathcal{T}_2$.

(c) Let $\mathcal{T}_X$ be the metric space topology for (X, d_X)

Let $\mathcal{T}_Y$ be the metric space topology for (Y, d_Y)

Let $\mathcal{T}_{X \times Y}$ be the product topology on $X \times Y$.

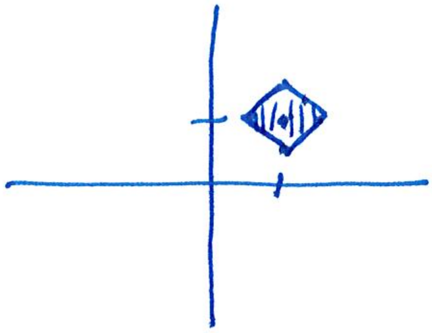
Then $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_{X \times Y}$.

<u>Examples</u> Let (X, d_X) be $\mathbb{R}$ with the metric $d_X(a_1,a_2) = |a_2 - a_1|$. Let (Y, d_Y) be $\mathbb{R}$ with the metric $d_Y(y_1,y_2) = |y_2 - y_1|$.

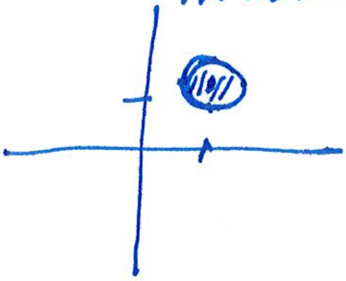
Then

$$d_1((x_1,y_1),(x_2,y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

$$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



$B_{\frac{1}{2}}((1,1))$ for $(\mathbb{R}^2, d_1)$



$B_{\frac{1}{2}}((1,1))$ for $(\mathbb{R}^2, d_2)$

Let $p \in \mathbb{R}_{\geq 1}$. Define $d_p : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ by

$$d_p((x_1, y_1), (x_2, y_2)) = \left(d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p\right)^{1/p}.$$

Define $d_\infty : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$ by

$$d_\infty((x_1, y_1), (x_2, y_2)) = \sup\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

<u>Proposition</u> (a) Show that d_p and d_∞ are metrics on $X \times Y$.

(b) Let $\mathcal{T}_p$ be the metric space topology on $(X \times Y, d_p)$

Let $\mathcal{T}_\infty$ be the metric space topology on $(X \times Y, d_\infty)$

Let $\mathcal{T}_X$ be the metric space topology on (X, d_X)

Let $\mathcal{T}_Y$ be the metric space topology on (Y, d_Y).

Let $\mathcal{T}_{X \times Y}$ be the product topology for $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.

Show that

$$\mathcal{T}_p = \mathcal{T}_\infty = \mathcal{T}_{X \times Y}.$$