Metric and Hilbert spaces, Lecture 14 — 24.08.2017 (1)
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Topologically equivalent metric spaces

Let X be a set. Let

$$d_1 : X \times X \to \mathbb{R}_{\geq 0} \quad \text{and} \quad d_2 : X \times X \to \mathbb{R}_{\geq 0}$$

be metrics on X.

The metric spaces (X, d_1) and (X, d_2) are topologically equivalent if

$$\mathcal{T}_{d_1} = \mathcal{T}_{d_2},$$

where $\mathcal{T}_{d_1}$ is the metric space topology on (X, d_1) and $\mathcal{T}_{d_2}$ is the metric space topology on (X, d_2).

The metric spaces (X, d_1) and (X, d_2) are Lipschitz equivalent if there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that

if $x, y \in X$ then $c_1 d_2(x,y) \leq d_1(x,y) \leq c_2 d_2(x,y)$.

Proposition If (X, d_1) and (X, d_2) are Lipschitz equivalent then (X, d_1) and (X, d_2) are topological equivalent.

Proof To show: $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.

To show: (a) $\mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2}$

(b) $\mathcal{T}_{d_2} \subseteq \mathcal{T}_{d_1}$

(a) To show: If $\mathcal{U} \in \mathcal{T}_{d_1}$ then $\mathcal{U} \in \mathcal{T}_{d_2}$.

Assume $\mathcal{U} \in \mathcal{T}_{d_1}$

To show: $\mathcal{U} \in \mathcal{T}_{d_2}$.

To show: If $x \in \mathcal{U}$ then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B^{d_2}_\varepsilon(x) \subseteq \mathcal{U}$.

Assume $x \in \mathcal{U}$.

To show: There exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B^{d_2}_\varepsilon(x) \subseteq \mathcal{U}$.

We know there exists $\delta \in \mathbb{R}_{>0}$ such that $B^{d_1}_\delta(x) \subseteq \mathcal{U}$.

Let $\varepsilon = c_2\delta$.

To show: $B^{d_2}_\varepsilon(x) \subseteq \mathcal{U}$.

To show: $B^{d_2}_\varepsilon(x) \subseteq B^{d_1}_\delta(x)$.

To show: If $y \in B^{d_2}_\varepsilon(x)$ then $y \in B^{d_1}_\delta(x)$.

Assume $y \in B^{d_2}_\varepsilon(x)$.

To show: $y \in B^{d_1}_\delta(x)$

To show: $d_1(y,x) < \delta$.

$$d_1(y,x) \le c_2 d_2(y,x) < c_2\varepsilon = \delta$$

Let (X,d) be a metric space. Define A. Ram

$b: X\times X \to \mathbb{R}_{\geq 0}$ by

$$b(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Then (a) b is a metric on X.

(b) (X,b) is bounded.

(c) $\mathcal{T}_b = \mathcal{T}_d$ where

$\mathcal{T}_b$ is the metric space topology on (X,b)

$\mathcal{T}_d$ is the metric space topology on (X,d).

Proof (a) To show: (aa) If $x\in X$ then $b(x,x)=0$.

(ab) If $x,y\in X$ and $b(x,y)=0$ then $x=y$.

(ac) If $x,y\in X$ then $b(x,y)=b(y,x)$

(ad) If $x,y,z\in X$ then $b(x,y)\leq b(x,z)+b(z,y)$.

(ad) Assume $x,y,z\in X$.

To show: $b(x,y)\leq b(x,z)+b(z,y)$.

$$b(x,z)+b(z,y) = \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$$

$$= \frac{d(x,z)+d(x,z)d(z,y)+d(z,y)d(x,z)+d(z,y)}{(1+d(x,z))(1+d(z,y))}$$

$$\geq \frac{d(x,y)+2d(x,z)d(z,y)}{1+d(x,z)+d(z,y)+d(x,z)d(z,y)}$$

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To show: $\delta(x,y) \le \delta(x,z) + \delta(z,y)$.

To show: $$\frac{d(x,y)}{1+d(x,y)} \le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$$

To show: $$d(x,y)(1+d(x,z))(1+d(z,y))$$
$$\le d(x,z)(1+d(x,y))(1+d(z,y))$$
$$+ d(z,y)(1+d(x,y))(1+d(x,z))$$

To show: $$d(x,y) + d(x,y)d(x,z) + d(x,y)d(z,y) + d(x,y)d(x,z)d(z,y)$$
$$\le d(x,z) + d(x,z)d(x,y) + d(x,z)d(z,y) + d(x,z)d(x,y)d(z,y)$$
$$+ d(z,y) + d(z,y)d(x,y) + d(z,y)d(x,z) + d(z,y)d(x,y)d(x,z).$$

To show: $$d(x,y) \le d(x,z) + d(x,z)d(z,y) + d(z,y) + d(z,y)d(x,z)$$
$$+ d(z,y)d(x,y)d(x,z).$$

$$d(x,y) \le d(x,z) + d(z,y)$$
$$\le d(x,z) + d(z,y) + d(x,z)d(z,y) + d(z,y)d(x,z)$$
$$+ d(z,y)d(x,y)d(x,z).$$

(b) To show: (X,b) is bounded.

To show: There exists $M \in \mathbb{R}_{>0}$ such that if $x,y \in X$ then $b(x,y) < M$.

Let $M = 1$.

To show: If $x,y \in X$ then $b(x,y) < 1$.

Assume $x,y \in X$.

To show: $b(x,y) < 1$.

$$b(x,y) = \frac{d(x,y)}{1+d(x,y)} < \frac{1+d(x,y)}{1+d(x,y)} = 1.$$

(c) To show: $\mathcal{T}_b = \mathcal{T}_d$.

To show: (ca) $\mathcal{T}_b \subseteq \mathcal{T}_d$

(cb) $\mathcal{T}_d \subseteq \mathcal{T}_b$.

<u>Claim</u>: Let $x \in X$. If $\delta \in \mathbb{R}_{>0}$ then there exist $\varepsilon \in \mathbb{R}_{>0}$ and $\gamma \in \mathbb{R}_{>0}$ such that

$$B_\gamma^d(x) \subseteq B_\varepsilon^b(x) \subseteq B_\delta^d(x).$$

Assume $x \in X$ and $\delta \in \mathbb{R}_{>0}$.

Let $\varepsilon = \dfrac{\delta}{1+\delta}$ and $\gamma = \cancel{\dfrac{\varepsilon}{1+\varepsilon}}\ \varepsilon$.

To show: (A) If $y \in B_\varepsilon^b(x)$ then $y \in B_\delta^d(x)$.

(B) If $z \in B_\gamma^d(x)$ then $z \in B_\varepsilon^b(x)$.

(A) Assume $y \in B_\varepsilon^b(x)$.

To show: $y \in B_\delta^d(x)$.

To show: $d(y,x) < \delta$.

$$d(y,x) = \frac{b(y,x)}{1-b(y,x)} \le \frac{b(y,x)}{1-\frac{\delta}{1+\delta}} = \frac{b(y,x)}{\frac{1}{1+\delta}}$$

$$= (1+\delta)\, b(y,x) \le (1+\delta)\frac{\delta}{1+\delta} = \delta.$$

(B) Assume $z \in B_\gamma^d(x)$

To show: $z \in B_\varepsilon^b(x)$.

To show: $b(z,x) < \varepsilon$

$$b(z,x) = \frac{d(z,x)}{1+d(z,x)} \le d(z,x) < \gamma = \varepsilon.$$