Metric and Hilbert spaces: Lecture 15

M&H Lect15
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<u>Proposition</u> Let (X,d) be a complete metric space. Let $A \subseteq X$.

If A is closed then A is complete.

<u>Proof</u> Assume A is closed.

To show: A is complete

To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then there exists $z \in A$ such that $\lim_{n\to\infty} a_n = z$.

Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in A.
Since X is complete then there exists $z \in X$ such that $\lim_{n\to\infty} a_n = z$.

We know

$$\bar{A} = \left\{ y \in X \;\middle|\; \begin{matrix}\text{there exists a sequence} \\ (a_1, a_2, \ldots) \text{ such that } y = \lim_{n\to\infty} a_n\end{matrix} \right\}.$$

So $z \in \bar{A}$.

Since A is closed then $A = \bar{A}$.

So $z \in A$

So A is complete//.

Theorem $\mathbb{R}_{\geq 0}$ is complete.

Proof sketch To show: If $(x_1, x_2, \dots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$ then there exists $z \in \mathbb{R}_{\geq 0}$ such that $z = \lim_{n\to\infty} x_n$.

Assume $(x_1, x_2, \dots)$ is a Cauchy sequence in $\mathbb{R}_{\geq 0}$

$$x_1 = a_1 . x_{11} x_{12} x_{13} \dots$$
$$x_2 = a_2 . x_{21} x_{22} x_{23} \dots$$
$$x_3 = a_3 . x_{31} x_{32} x_{33} \dots$$
$$\vdots$$

with $a_j \in \mathbb{Z}_{\geq 0}$ and $x_{ij} \in \{0, 1, \dots, 9\}$.

For $k \in \mathbb{Z}_{>0}$ let ℓ_k be such that

if $m, n \in \mathbb{Z}_{\geq \ell_k}$ then $d(x_m, x_n) < 10^{-k}$.

Let
$$z_{\leq k} = (x_{\ell_k})_{\leq k}.$$

Then $z_{\leq k}$ is the first k decimal places of an element $z \in \mathbb{R}_{\geq 0}$

and $\lim_{n\to\infty} x_n = z$. //

<u>Theorem</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

(a) If E is connected then $f(E)$ is connected

(b) If E is compact then $f(E)$ is compact.

<u>Theorem</u> Let $X = \mathbb{R}$ with the standard metric.

(a) $\mathbb{R}$ is a complete metric space.

(b) $E \subseteq \mathbb{R}$ is connected if and only if E is an interval.

(c) $E \subseteq \mathbb{R}$ is compact if and only if E is closed and bounded.

(d) $E \subseteq \mathbb{R}$ is connected and compact if and only if there exists $m, M \in \mathbb{R}$ such that $E = [m, M]$.

<u>Theorem</u> (Intermediate value theorem).
Let $a, b \in \mathbb{R}$ with $a < b$.

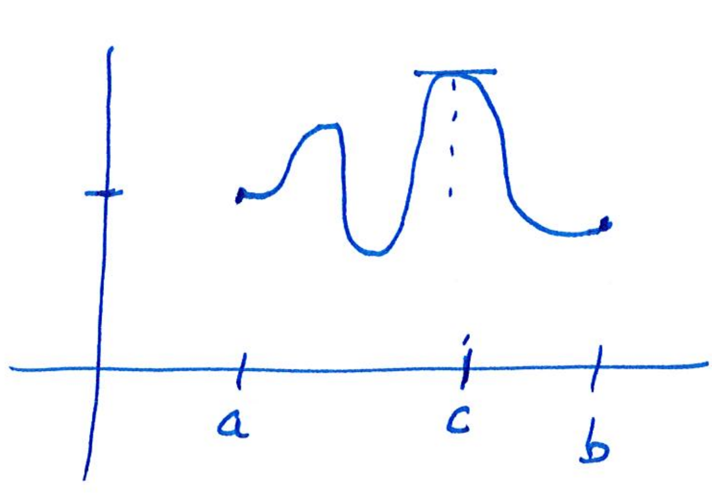
(a) If $f : [a,b] \to \mathbb{R}$ is a continuous function and $w \in (f(a), f(b))$ then there exists $c \in (a,b)$ such that $f(c) = w$.

(b) If $f : [a,b] \to \mathbb{R}$ is a continuous function then there exist $m, M \in \mathbb{R}$ such that

$$f([a,b]) = [m, M].$$

<u>Theorem</u> (Rolle's theorem) Let $a, b \in \mathbb{R}$ with $a < b$.
If $f : [a,b] \to \mathbb{R}$ is a function such that f is continuous and $f' : (a,b) \to \mathbb{R}$ exists and $f(a) = f(b)$ then there exists

$$c \in (a,b) \text{ such that } f'(c) = 0.$$

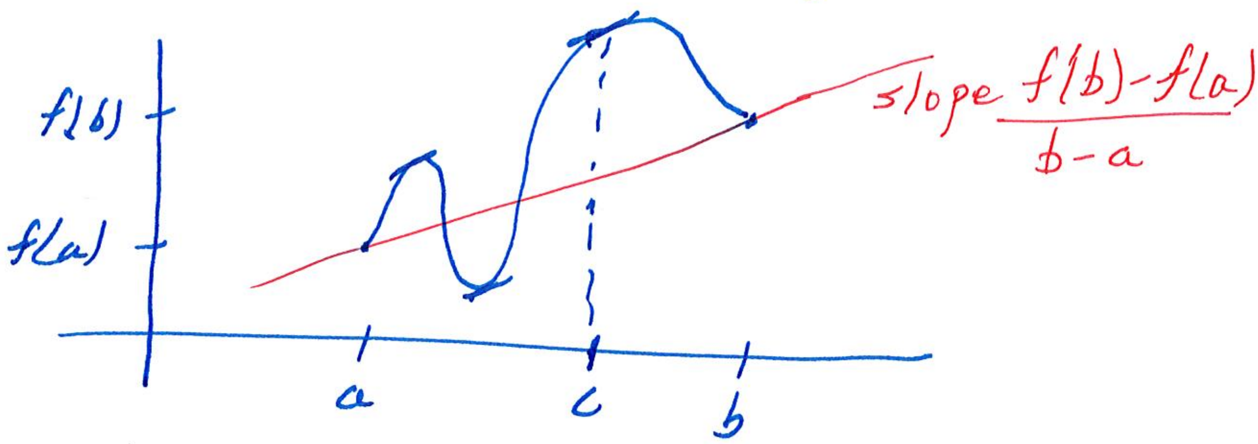


<u>Theorem</u> (Mean value theorem) Let $a, b \in \mathbb{R}$ with $a < b$. If $f : [a,b] \to \mathbb{R}$ is a function such that

f is continuous and $f' : (a,b) \to \mathbb{R}$ exists

then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(c)(b-a)$$



<u>Theorem</u> (Taylor's theorem) Let $a, b \in \mathbb{R}$ with $a < b$. Let $N \in \mathbb{Z}_{>0}$. If $f : [a,b] \to \mathbb{R}$ is a function such that

$f^{(N)} : [a,b] \to \mathbb{R}$ is continuous and

$f^{(N+1)} : (a,b) \to \mathbb{R}$ exists

then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2$$
$$+ \cdots + \frac{1}{N!} f^{(N)}(a)(b-a)^N$$
$$+ \frac{1}{(N+1)!} f^{(N+1)}(c)(b-a)^{N+1}$$