Metric and Hilbert Spaces: Lecture 17 31.08.2017 ①

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Function spaces

Let (X, d_X) and (Y, d_Y) be metric spaces. Let

$$F = \{\text{functions } f: X \to Y\}.$$

Define $d_\infty : F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$d_\infty(f, g) = \sup\{ d_Y(f(x), g(x)) \mid x \in X \}$$

Let $f_1, f_2, \ldots$ be a sequence of functions. Let $f \in F$.

The sequence $(f_1, f_2, \ldots)$ converges pointwise to f if $f_1, f_2, \ldots$ satisfy:

if $x \in X$ then $\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0$.

The sequence $(f_1, f_2, \ldots)$ converges uniformly to f if $f_1, f_2, \ldots$ satisfy:

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Function spaces that are normed vector spaces

Let X be a set and

$$F = \{\text{functions } f: X \to \mathbb{R}\}.$$

with $\|f\|_\infty = \sup\{ |f(x)| \mid x \in \mathbb{R}\}$.

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Then F is a $\mathbb{R}$ vector space with

$f+g$ given by $(f+g)(x) = f(x) + g(x)$

cf given by $(cf)(x) = cf(x)$

<u>Special cases</u>: $X = \mathbb{Z}_{[1,n]}$ and $X = \mathbb{Z}_{>0}$.

$$\mathbb{R}^n = \{\text{functions } f: \mathbb{Z}_{[1,n]} \to \mathbb{R}\}$$

$$= \{\text{functions } x: \{1,2,\ldots,n\} \to \mathbb{R}\}$$

$$\begin{matrix} 1 \longmapsto x_1 \\ 2 \longmapsto x_2 \\ \vdots \end{matrix}$$

$$= \{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\} = \mathbb{R}^n$$

and $\|x\|_\infty = \sup\{|x_1|, \ldots, |x_n|\}$.

$$\ell^\infty = \{\text{functions } f: \mathbb{Z}_{>0} \to \mathbb{R} \mid \|f\|_\infty < \infty\}$$

$$= \left\{ (x_1, x_2, \ldots) \;\middle|\; \begin{matrix} x_i \in \mathbb{R} \text{ and} \\ \sup\{|x_1|, |x_2|, \ldots\} < \infty \end{matrix} \right\}$$

<u>ℓ^p-spaces</u> Let $p \in \mathbb{R}_{\geq 1}$

For $x = (x_1, x_2, \ldots)$ define

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots)^{1/p} = \Big(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p\Big)^{1/p}.$$

$$\ell^p = \{x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|x\|_p < \infty\}.$$

Exercise: Show that the ℓ^p are normed vector spaces

<u>$B(V,W)$</u>, <u>bounded linear transformations</u>

Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed vector spaces. Let

$T: V \to W$ be a linear transformation.

Define

$$\|T\| = \sup\left\{ \frac{\|Tv\|_W}{\|v\|_V} \;\middle|\; v \in V \text{ and } v \neq 0 \right\}$$

<u>Exercise</u>: Show that the space $B(V,W)$ of <u>bounded linear transformations from V to W</u>

$$B(V,W) = \left\{ T: V \to W \;\middle|\; \begin{array}{l} T \text{ is a linear transformation} \\ \|T\| < \infty \end{array} \right\}$$

is a normed vector space.

<u>Dual spaces</u>

$(\mathbb{R}, |\ |)$ is a normed vector space.

Let $(V, \|\ \|)$ be a normed vector space.

The <u>dual of V</u> is

$$V^* = B(V, \mathbb{R}) = \left\{ T: V \to \mathbb{R} \;\middle|\; \begin{array}{l} T \text{ is linear transf.} \\ \text{and } \|T\| < \infty \end{array} \right\}$$

$$= \left\{ \varphi: V \to \mathbb{R} \;\middle|\; \begin{array}{l} \varphi \text{ is a linear transformation} \\ \text{and } \|\varphi\| < \infty \end{array} \right\}$$

Elements of V^* are bounded linear functionals on V.

$(V^*, \| \|)$ is a normed vector space.