Metric and Hilbert spaces Lecture 18

Inner product spaces

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

For $\mathbb{K} = \mathbb{C}$,
$$\overline{a+bi} = a - bi, \quad \text{where } a, b \in \mathbb{R},\ i^2 = -1.$$
For $\mathbb{K} = \mathbb{R}$
$$\bar{a} = a, \quad \text{where } a \in \mathbb{R}.$$

An <u>inner product space</u> is a vector space V with a function $\langle , \rangle : V \times V \to \mathbb{K}$ such that

(a) If $v_1, v_2, v_3 \in V$ then $\langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$.

(b) If $c \in \mathbb{K}$ and $v_1, v_2 \in V$ then $\langle cv_1, v_2\rangle = c\langle v_1, v_2\rangle$.

(c) If $v_1, v_2, v_3 \in V$ then $\langle v_1, v_2 + v_3\rangle = \langle v_1, v_2\rangle + \langle v_1, v_3\rangle$.

(d) If $c \in \mathbb{K}$ and $v_1, v_2 \in V$ then $\langle v_1, cv_2\rangle = \bar{c}\langle v_1, v_2\rangle$.

(e) If $v_1, v_2 \in V$ then $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$.

(f) If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$.

(g) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

Let $(V, \langle\ \rangle)$ be a normed vector space.

The <u>length norm</u> on V is $\|\ \| : V \to \mathbb{R}_{\geq 0}$ given by
$$\|v\| = \sqrt{\langle v, v\rangle}.$$

<u>Proposition</u> Let $(V, \langle\,\rangle)$ be an inner product space.

(a) (Pythagorean theorem) If $x, y \in V$ and $\langle x, y\rangle = 0$ then $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

(b) (Parallelogram Law) If $x, y \in V$ then
$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(c) (Cauchy-Schwarz) If $x, y \in V$ then
$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

(d) (triangle inequality) If $x, y \in V$ then
$$\|x+y\| \le \|x\| + \|y\|.$$

<u>Proof</u> (c) Assume $x, y \in V$. Let $W = \text{span}\{x, y\}$.

<u>Case 0</u>: $\dim(W) = 0$. Then $x = 0$ and $y = 0$ and
$$\langle x, y\rangle = 0 = \|x\| \cdot \|y\|.$$

<u>Case 1</u>: $\dim(W) = 1$. Then there exists $c \in \mathbb{K}$ such that $y = cx$ and
$$|\langle x, y\rangle| = |\langle x, cx\rangle| = |c| \cdot \|x\|^2 = \|x\| \cdot \|cx\| = \|x\| \cdot \|y\|.$$

<u>Case 2</u>: $\dim(W) = 2$. Then $\{x, y\}$ is a basis of W.

Another basis is $\{v_1, v_2\}$ where

$$v_1 = \frac{x}{\|x\|} \quad \text{and} \quad v_2 = \frac{y - \langle v_1, y\rangle v_1}{\| y - \langle v_1, y\rangle v_1 \|}$$

Write $x = a_1 v_1 + a_2 v_2$ and $y = b_1 v_1 + b_2 v_2$.

Then

$$|\langle x, y\rangle| = a_1 b_1 + a_2 b_2 \quad \text{and}$$

$$\|x\|^2 = a_1^2 + a_2^2 \quad \text{and} \quad \|y\|^2 = b_1^2 + b_2^2.$$

Then

$$\begin{aligned}
\|\langle x, y\rangle\| &= (a_1 b_1 + a_2 b_2)^2 \\
&\le (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2 \\
&= a_1^2 b_1^2 + 2 a_1 a_2 b_1 b_2 + a_2^2 b_2^2 + a_1^2 b_2^2 - 2 a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\
&= a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_2^2 \\
&= (a_1^2 + a_2^2)(b_1^2 + b_2^2) = \|x\|^2 \|y\|^2.
\end{aligned}$$

So $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$.

(d) Using (c)

$$\begin{aligned}
\mathrm{Re}\langle x, y\rangle &\le \sqrt{(\mathrm{Re}\langle x, y\rangle)^2} \le \sqrt{\mathrm{Re}(\langle x, y\rangle)^2 + (\mathrm{Im}\langle x, y\rangle)^2} \\
&= |\langle x, y\rangle| \le \|x\| \cdot \|y\|
\end{aligned}$$

so that

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle) \\
&\le \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2.
\end{aligned}$$

So $\|x+y\| \le \|x\| + \|y\|$. //.

<u>Proposition</u> Let $(V, \| \|)$ be a normed vector space.

Assume $\| \|$ satisfies:

if $x, y \in V$ then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

If $\mathbb{K} = \mathbb{R}$ define $\langle\ \rangle : V \times V \to \mathbb{R}$ by

$$\langle x, y \rangle = \tfrac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

If $\mathbb{K} = \mathbb{C}$ define $\langle\ \rangle : V \times V \to \mathbb{C}$ by

$$\langle x, y \rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

Then $(V, \langle\ \rangle)$ is an inner product space such that

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

A <u>quadratic form on V</u> is a function $Q : V \to \mathbb{K}$ such that

(a) If $v \in V$ and $c \in \mathbb{K}$ then $Q(cv) = c^2 Q(v)$

(b) The map $\langle , \rangle : V \times V \to \mathbb{K}$ given by

$$\langle x, y \rangle = Q(x+y) - Q(x) - Q(y)$$

satisfies

if $x_1, x_2, x_3 \in V$ then $\langle x_1 + x_2, x_3 \rangle = \langle x_1, x_3 \rangle + \langle x_2, x_3 \rangle$

and $\langle x_1, x_2 + x_3 \rangle = \langle x_1, x_2 \rangle + \langle x_1, x_3 \rangle$

and if $x_1, x_2 \in V$ and $c \in \mathbb{K}$ then

$\langle cx_1, x_2 \rangle = \langle x_1, x_2 \rangle \cdot c$ and $\langle x_1, cx_2 \rangle = c\langle x_1, x_2 \rangle$.