Metric and Hilbert Spaces Lecture 1

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Web page: search for "Arun Ram".

Housekeeping / Consultation / Tutorials / Assignments

Resources: Lecture Capture / Notes etc.

Math: Language / Vocabulary / Grammar

Grammar = Proof machine

Plagiarism / Referencing

<u>Convergence</u>: Chapter 6.

A <u>metric space</u> is a set X with a function $d: X\times X \to \mathbb{R}_{\geq 0}$ such that

(a) if $x\in X$ then $d(x,x)=0$,

(b) if $x,y\in X$ and $d(x,y)=0$ then $x=y$,

(c) If $x,y\in X$ then $d(x,y)=d(y,x)$

(d) If $x,y,z\in X$ then

$$d(x,y)\leq d(x,z)+d(z,y)$$

Seven theorems in one box

Let (X, d) be a metric space

$$\begin{array}{ccccc}
\text{cover compact} & \Rightarrow & \text{ball compact} & \Rightarrow & \text{bounded} \\
\Downarrow & & \Leftarrow & & + \\
\text{sequentially compact} & \Rightarrow & \text{Cauchy compact} & \Rightarrow & \text{closed}
\end{array}$$

<u>Vocabulary</u> Let (X, d) be a metric space.

~~Let~~ Let $A \subseteq X$. Let $\mathcal{T}$ be the metric space topology on X.

A is <u>cover compact</u> if A satisfies

if $\mathcal{S} \subseteq \mathcal{T}$ and $\left(\bigcup_{U \in \mathcal{S}} U\right) \supseteq A$

then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{S}$ such that

$$U_1 \cup U_2 \cup \cdots \cup U_\ell \supseteq A.$$

A is <u>sequentially compact</u> if A satisfies

if $(a_1, a_2, \ldots)$ is a sequence in A

then there exists $z \in A$ such that

z is a cluster point of $(a_1, a_2, \ldots)$

A is <u>ball compact</u> if A satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exist $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that

$$B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_\ell) \supseteq A.$$

A is <u>Cauchy compact</u> if A satisfies:

if $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then there exists $z \in A$ such that

z is a limit point of $(a_1, a_2, \ldots)$.

A is <u>bounded</u> if A satisfies:

There exists $M \in \mathbb{R}_{>0}$ and $x \in X$ such that

$$B_M(x) \supseteq A$$

A is <u>closed</u> if A satisfies:

if $(a_1, a_2, \ldots)$ is a sequence in A and $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then $z \in A$.