Proposition Let $(V, \|\ \|)$ and $(W, \|\ \|_W)$ be normed vector spaces. Let $T: V \to W$ be a linear transformation. The following are equivalent.

(a) T is bounded.

(b) T is continuous.

(c) T is uniformly continuous.

Proof (c) $\Rightarrow$ (b) is already proved.

(a) $\Rightarrow$ (c): Assume $T \in B(V, W)$

To show: T is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in V$ and $d(x,y) < \delta$ then $d(T(x), T(y)) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in V$ and $d(x,y) < \delta$ then $d(T(x), T(y)) < \varepsilon$

Let $\delta = \dfrac{\varepsilon}{\|T\|}$.

To show: If $x, y \in V$ and $d(x,y) < \delta$ then $d(T(x), T(y)) < \varepsilon$.

Assume $x, y \in V$ and $d(x,y) < \delta$.

To show: $d(T(x), T(y)) < \varepsilon$.

$$d(T(x), T(y)) = \|T(x) - T(y)\| = \|T(x-y)\|$$
$$\leq \|T\| \cdot \|x-y\| = \|T\| \, d(x,y)$$
$$\leq \|T\| \cdot \delta = \varepsilon.$$

So T is uniformly continuous.

(b) $\Rightarrow$ (a) To show: If T is continuous then T is bounded.

Assume T is continuous.

To show: $\|T\| < \infty$.

To show: There exists $C \in \mathbb{R}_{>0}$ such that if $u \in V$ then $\|Tu\| \leq C\|u\|$.

Since T is continuous then T is continuous at 0.

So there exists $\delta \in \mathbb{R}_{>0}$ such that if $\|x\| < \delta$ then $\|Tx\| < 1$.

Let $C = \frac{2}{\delta}$

To show: If $u \in V$ then $\|Tu\| \leq C\|u\|$.

Assume $u \in V$.

To show: $\|Tu\| \leq C\|u\|$.

To show: $\frac{\|Tu\|}{\|u\|} \le C$

$$\frac{\|Tu\|}{\|u\|} = \left\|T\frac{u}{\|u\|}\right\| = \frac{2}{\delta}\left\|T\left(\frac{\delta}{2}\frac{u}{\|u\|}\right)\right\|$$

$$\le \frac{2}{\delta}\cdot 1 = C.$$

So T is bounded.

(c) $\Rightarrow$ (b): Assume T is uniformly continuous.

Then T satisfies:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $v \in V$ and $v' \in V$ and $\|v - v'\| < \delta$ then $\|T(v) - T(v')\| < \varepsilon$.

Thuse if $v \in V$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $v' \in V$ and $\|v - v'\| < \delta$ then $\|T(v) - T(v')\| < \varepsilon$.

So $T: V \to W$ is continuous. //

Let $(V, \langle\,\rangle)$ be an inner product space and let $S \subseteq V$. The <u>orthogonal to S</u> is

$$S^\perp = \{v \in V \mid \text{If } w \in S \text{ then } \langle v, w\rangle = 0\}$$

<u>Proposition</u>: Let $(V, \langle\,\rangle)$ be an inner product space. Let $W \subseteq V$.

(a) $W^\perp$ is a closed subspace of V.

(b) If V is a Hilbert space ~~and~~ then W is a closed subspace of V if and only if $V = W \oplus W^\perp$.

<u>Proof of (a)</u> Let

$$\begin{aligned} \Phi : V &\to W^* \\ v &\mapsto \Phi_v \end{aligned} \quad \text{where}$$

$$\begin{aligned} \Phi_v : W &\to \mathbb{R} \\ w &\mapsto \langle v, w\rangle. \end{aligned} \quad \text{Then}$$

$$\begin{aligned} W^\perp &= \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w\rangle = 0\} \\ &= \{v \in V \mid \Phi_v = 0\} \\ &= \ker \Phi = \Phi^{-1}(\{0\}). \end{aligned}$$

Since $\{0\}$ is closed in W^* $\left(\{0\}^c = \left\{ x \in W^* \,\middle|\, \begin{array}{l} \text{there exists} \\ \varepsilon \in \mathbb{R}_{>0} \text{ with} \\ B_\varepsilon(x) \subseteq \{0\}^c \end{array} \right\} \right)$ and Φ is continuous then

$$W^\perp = \Phi^{-1}(\{0\}) \text{ is closed in } W$$