Metric and Hilbert spaces: Lecture 23 14.09.2017 A. Ram UniMelb (1)

Duals and adjoints

Let $(V, \| \|)$ be a normed $\mathbb{K}$-vector space.

The <u>dual of V</u>, or the space of <u>bounded</u> <u>linear</u> <u>functionals</u> <u>on V</u>, is

$$V^* = B(V, \mathbb{K}) = \left\{ \varphi: V \to \mathbb{K} \ \middle|\ \begin{matrix} \varphi \text{ is a linear transf.} \\ \|\varphi\| < \infty \end{matrix} \right\}$$

Let $(V, \| \|_V)$ and $(W, \| \|_W)$ be normed vector spaces. Let $T: V \to W$ be a linear transformation. The <u>adjoint</u> <u>of T</u> is the linear transformation

$$T^*: W^* \to V^* \quad \text{given by} \quad (T^*\varphi)(v) = \varphi(T(v))$$

<u>Theorem</u> (Riesz representation theorem). Let $(H, \langle\, \rangle)$ be a Hilbert space. Then

$$\begin{matrix} \Phi: H \to H^* \\ x \mapsto \Phi_x \end{matrix} \quad \text{where} \quad \begin{matrix} \Phi_x: H \to \mathbb{K} \\ h \mapsto \langle h, x \rangle. \end{matrix}$$

is a skew linear bijective isometry and $\|\Phi\| = 1$.

<u>Proof</u> To show:
(a) Φ is skew linear.
(b) Φ is an isometry
(c) $\|\Phi\| = 1$
(d) Φ is injective
(e) Φ is surjective.

(a) To show: (aa) If $a, b \in H$ then $\Phi_{a+b} = \Phi_a + \Phi_b$

(ab) If $a \in H$ and $c \in \mathbb{K}$ then $\Phi_{ca} = \bar{c}\,\Phi_a$.

(aa) Assume $a, b \in H$.

To show: $\Phi_{a+b} = \Phi_a + \Phi_b$.

To show: If $h \in H$ then $\Phi_{a+b}(h) = \Phi_a(h) + \Phi_b(h)$.

Assume $h \in H$.

To show: $\Phi_{a+b}(h) = \Phi_a(h) + \Phi_b(h)$

$$\Phi_{a+b}(h) = \langle h, a+b\rangle = \langle h, a\rangle + \langle h, b\rangle = \Phi_a(h) + \Phi_b(h)$$

(ab) Assume $a \in H$ and $c \in \mathbb{K}$.

To show: $\Phi_{ca} = \bar{c}\,\Phi_a$

To show: If $h \in H$ then $\Phi_{ca}(h) = \bar{c}\,\Phi_a(h)$.

Assume $h \in H$.

To show: $\Phi_{ca}(h) = \bar{c}\,\Phi_a(h)$

$$\Phi_{ca}(h) = \langle h, ca\rangle = \bar{c}\langle h, a\rangle = \bar{c}\,\Phi_a(h).$$

(b) To show: Φ is an isometry.

To show: If $x \in H$ then $\|\Phi_x\| = \|x\|$.

Assume $x \in H$.

To show: (ba) $\|\Phi_x\| \le \|x\|$

(bb) $\|\Phi_x\| \ge \|x\|$.

(6a) Assume $h \in H$. By Cauchy-Schwarz,

$$\|\Phi_x(h)\| = |\langle h, x\rangle| \le \|h\|\cdot\|x\|.$$

So $\|\Phi_x\| \le \|x\|$

(6b) Since

$$\|\Phi_x(x)\| = |\langle x, x\rangle| = \|x\|^2 = \|x\|\cdot\|x\|$$

then $\|\Phi_x\| \ge \|x\|$.

So $\|\Phi_x\| = \|x\|$.

So Φ is an isometry.

(c) To show: $\|\Phi\| = 1$.

Using that $\|\Phi_x\| = \|x\|$ from (b),

$$\|\Phi\| = \sup\left\{ \frac{\|\Phi_x\|}{\|x\|} \mid x \in H, x \ne 0 \right\} = \sup\{1\} = 1.$$

(d) To show: Φ is injective.

To show: If $a, b \in H$ and $\Phi_a = \Phi_b$ then $a = b$.

Assume $a, b \in H$ and $\Phi_a = \Phi_b$.

To show: $a = b$.

To show: $\|a-b\| = 0$.

$$\|a-b\| = \|\Phi_{a-b}\| = \|\Phi_a - \Phi_b\| = \|0\| = 0.$$

So $a = b$

So Φ is injective.

(e) To show: Φ is surjective.

To show: If $\varphi \in H^*$ then there exists $a \in H$ such that $\varphi = \Phi_a$.

Assume $\varphi \in H^*$.

To show: There exists $a \in H$ such that $\varphi = \Phi_a$.

Case 1: $\varphi = 0$. Then $\varphi = \Phi_0$.

Case 2: $\varphi \neq 0$.

Since φ is bounded then φ is continuous.

Since $\{0\}$ is closed in $\mathbb{K}$ then

$$\ker\varphi = \varphi^{-1}(\{0\}) \text{ is closed in } H.$$

By the orthogonal decomposition theorem, since $\ker\varphi$ is a closed subspace of H then

$$H = \ker\varphi \oplus (\ker\varphi)^\perp.$$

Let $b \in (\ker\varphi)^\perp$ with $b \neq 0$ and let

$$a = \frac{\overline{\varphi(b)}}{\|b\|^2}\, b$$

To show: If $h \in H$ then $\varphi(h) = \Phi_a(h)$.

Assume $h \in H$.

$$h = \left(h - \frac{\varphi(h)}{\varphi(a)}a\right) + \frac{\varphi(h)}{\varphi(a)}a \text{ with } h - \frac{\varphi(h)}{\varphi(a)}a \in \ker\varphi.$$

To show: $\varphi(h) = \Psi_a(h)$.

Since

$$\langle a,a\rangle = \left\langle \frac{\overline{\varphi(b)}}{\|b\|^2}b, \frac{\overline{\varphi(b)}}{\|b\|^2}b\right\rangle = \frac{\overline{\varphi(b)}\varphi(b)}{\|b\|^2\|b\|^2}\langle b,b\rangle$$

$$= \frac{\overline{\varphi(b)}}{\|b\|^2}\varphi(b) = \varphi(a)$$

and since $a \in (\ker\varphi)^\perp$ then

$$\Psi_a(h) = \langle h,a\rangle = \left\langle \left(h - \frac{\varphi(h)}{\varphi(a)}a\right) + \frac{\varphi(h)}{\varphi(a)}a, a\right\rangle$$

$$= 0 + \frac{\varphi(h)}{\varphi(a)}\langle a,a\rangle = \varphi(h).$$

So $\Psi_a = \varphi$.

So Ψ is surjective. //