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# Metric and Hilbert Spaces: Lecture 28

<u>Proposition</u> Let $(H, \langle\,\rangle)$ be a Hilbert space and let $W$ be a subspace of $H$. Let $\overline{W}$ be the closure of $W$. Then

$$H = \overline{W} \oplus \overline{W}^{\perp}.$$

<u>Theorem</u> Let $(V, \langle\,\rangle)$ be an inner product space and let $T: V \to V$ be a bounded self adjoint compact operator. Then there exists an eigenvector $v \in V$ for $T$ (and $Tv = \|T\| v$).

Let $V$ be a vector space and let $T: V \to V$ be a linear transformation.

A <u>$T$-submodule of $V$</u> is a subspace $W \subseteq V$ such that

if $w \in W$ then $Tw \in W$

<u>Examples</u>

(1) Let $v$ be an eigenvector of $T: V \to V$. Then $W = \mathbb{K}\text{-span}\{v\}$ (a 1-dimensional space) is a $T$-submodule.

(2) The subspace $\ker(T) = W$ is a $T$-submodule.

(3) Let $\lambda \in \mathbb{K}$. The $\lambda$-eigenspace

$$X_\lambda = \{ v \in V \mid Tv = \lambda v \} \text{ is a } T\text{-submodule.}$$

$\ker(T) = X_0$.

(4) Let $\sigma_p(T) = \{ \lambda \in \mathbb{K} \mid X_\lambda \neq 0 \}$ (the point spectrum).
Then

$$W = \left( \bigoplus_{\lambda \in \sigma_p(T)} X_\lambda \right) \text{ is a } T\text{-submodule.}$$

(5) Let $W \subseteq V$ be a $T$-submodule of $V$.
Assume $T: V \to V$ is bounded.
Then $T: V \to V$ is continuous.
If $z \in \overline{W}$ then there exists $(w_1, w_2, \ldots)$ with

$$\lim_{n \to \infty} w_n = z.$$

So $Tz = T\left( \lim_{n \to \infty} w_n \right) = \lim_{n \to \infty} (T w_n) \in \overline{W}$.

So $\overline{W}$ is a $T$-submodule of $V$.

(6) Let $W \subseteq V$ be a $T$-submodule of $V$.
Assume $V$ is an inner product space
and $T: V \to V$ is a self adjoint operator.

Let ~~z~~ $z \in W^\perp$.

To show: $Tz \in W^\perp$

To show: If $w \in W$ then $\langle Tz, w\rangle = 0$.

Assume $w \in W$. Then

$\langle Tz, w\rangle = \langle z, Tw\rangle = 0$, since $Tw \in W$ and $z \in W^\perp$.

So $Tz \in W^\perp$.

So $W^\perp$ is a $T$-submodule of $V$.

<u>Proposition</u> Let $(H, \langle\,\rangle)$ be a Hilbert space and let $T: H \to H$ be a bounded selfadjoint compact operator. Let

$$W = \Big(\bigoplus_{\lambda \in \sigma_p(T)} X_\lambda\Big). \quad \text{Then} \quad H = \overline{W}.$$

<u>Proof</u> To show: $H = \overline{W}$.

We know: $H = \overline{W} \oplus \overline{W}^\perp$ and $T: \overline{W}^\perp \to \overline{W}^\perp$.

Claim: $T: \overline{W}^\perp \to \overline{W}^\perp$ is a compact operator.

Assume $(z_1, z_2, \ldots)$ is a sequence in $\overline{W}^\perp$ with $\|z_i\| \le 1$

Then $(Tz_1, Tz_2, \ldots)$ is a sequence in $\overline{W}^\perp$

Since $(Tz_1, Tz_2, \ldots)$ is a sequence in $H$ and $T: H \to H$ then $(Tz_1, Tz_2, \ldots)$ has a cluster point $z$.

So there is a subsequence $(Tz_{n_1}, Tz_{n_2}, \ldots)$

such that $\lim_{k\to\infty} Tz_{n_k} = z$.

So $z \in \overline{(W^\perp)}$. Since $W^\perp$ is closed $z \in W^\perp$.

So $(Tz_1, Tz_2, \ldots)$ has a cluster point in $W^\perp$.

So $T: W^\perp \to W^\perp$ is a compact operator.

To show: $W^\perp = 0$. $W^\perp \not\subseteq \ker T$ then $T: W^\perp \to W^\perp$ is not $0$.
Assume $W^\perp \neq 0$. Since
Since $T: W^\perp \to W^\perp$ is a bounded self adjoint compact operator there exists nonzero $z \in W^\perp$ with is an eigenvector for $T$.

Since $Tz = \lambda z$, then $z \in X_\lambda$ and $z \in W$.

So $z \in W \cap W^\perp$.

So $z = 0$, which is a contradiction to $z \neq 0$.

So $W^\perp = 0$.

So $H = \overline{W}$. //

<u>Theorem</u> Let $(H, \langle\, \rangle)$ be a Hilbert space with a countable dense set (separable). Let $T: H \to H$ be a bounded self adjoint compact operator. Then there exists an

orthonormal (topological) basis of $H$ consisting of eigenvectors of $T$.

Proof We know: $H = \overline{\ker(T) \oplus \left(\bigoplus_{\substack{\lambda\in\sigma_p(T)\\ \lambda\neq 0}} X_\lambda\right)}$

and $X_\lambda$ is finite dimension for $\lambda\in\sigma_p(T)$ with $\lambda\neq 0$.

Let $\mathcal{B}_0$ be an orthonormal basis of $\ker(T)$ constructed by Gram-Schmidt from a countable basis $(k_1, k_2, \ldots)$ of $\ker(T)$.

Let $\mathcal{B}_\lambda$ be an orthonormal basis of $X_\lambda$.

Then $$\mathcal{B} = \mathcal{B}_0 \sqcup \left(\bigsqcup_{\substack{\lambda\in\sigma_p(T)\\ \lambda\neq 0}} \mathcal{B}_\lambda\right)$$

is an orthonormal basis of $H$ since $X_\lambda \perp X_\gamma$ if $\lambda\neq\gamma$ and $H = \overline{\ker(T)\oplus\left(\oplus X_\lambda\right)}$. //