Let $V = \mathbb{C}^n$ with the standard inner product

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$$

Let $B \in M_n(\mathbb{C})$ and let

$$A = B^* B \quad \text{where} \quad B^* = \bar{B}^t.$$

Then $$A^* = (B^*B)^* = B^*(B^*)^* = B^*B = A$$

so that A is self-adjoint.

The orthonormal basis $(e_1, e_2, \ldots, e_n)$ with

$$e_i = (0, 0, \ldots 0, \overset{i\text{th}}{1}, 0, \ldots 0) \text{ is the favourite.}$$

Let $(a_1, a_2, \ldots, a_n)$ be an orthonormal basis of eigenvectors of A,

$$Aa_1 = \lambda_1 a_1, \quad Aa_2 = \lambda_2 a_2, \quad \ldots, \quad Aa_n = \lambda_n a_n$$

Let $K \in M_n(\mathbb{C})$ be the change of basis matrix from $(e_1, \ldots, e_n)$ to $(a_1, \ldots, a_n)$. Then

$$\delta_{ij} = \langle a_i, a_j \rangle = \langle Ke_i, Ke_j \rangle = \langle e_i, K^*Ke_j \rangle$$

so that

$$K^*K = 1 \quad \text{and} \quad KA\cancel{K}^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

(a) Assume $x \in V$. Then

$$\|Kx\|^2 = \langle Kx, Kx\rangle = \langle x, K^*Kx\rangle = \langle x, x\rangle$$
$$= \|x\|^2.$$

So $\|Kx\| = \|x\|$.

(b) To show: $\|KAK^{-1}\| = \|A\|$

To show: (ba) $\|KAK^{-1}\| \le \|A\|$

(bb) $\|KAK^{-1}\| \ge \|A\|$.

(ba) Let $v \in V$. Then

$$\|KAK^{-1}v\| = \|AK^{-1}v\| \le \|A\|\,\|K^{-1}v\| = \|A\|\cdot\|v\|.$$

So $\|KAK^{-1}\| \le \|A\|$.

(bb) Using (ba),

$$\|K^{-1}(KAK^{-1})K\| \le \|KAK^{-1}\|$$

giving $\|A\| \le \|KAK^{-1}\|$

So $\|KAK^{-1}\| = \|A\|$.

(c) To show: $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_{\ge 0}$.

Let $x \in V$ with $Ax = \lambda x$. Then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle = \langle x, \lambda x\rangle = \bar{\lambda}\|x\|^2.$$

So $\bar{\lambda} = \dfrac{\|Bx\|^2}{\|x\|^2}$. So $\lambda \in \mathbb{R}_{\ge 0}$.

(d) To show: $\|B\| = \sqrt{\gamma}$ where $\gamma = \max\{\lambda_1, \ldots, \lambda_n\}$.

To show: (da) $\|B\| \geq \sqrt{\gamma}$

(bb) $\|B\| \leq \sqrt{\gamma}$.

(da) Since $\dfrac{\|Ba_i\|}{\|a_i\|} = \sqrt{\lambda_i}$ then $\|B\| \geq \sqrt{\lambda_i}$.

So $\|B\| \geq \sqrt{\gamma}$

(db) Let $x \in V$ and write $x = c_1 a_1 + \cdots + c_n a_n$.

Then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^* Bx\rangle$$

$$= \langle x, A(c_1a_1 + \cdots + c_na_n)\rangle$$

$$= \langle c_1a_1 + \cdots + c_na_n, c_1\lambda_1 a_1 + \cdots + c_n\lambda_n a_n\rangle$$

$$= \lambda_1|c_1|^2 + \lambda_2|c_2|^2 + \cdots + \lambda_n|c_n|^2$$

$$\leq \gamma(|c_1|^2 + \cdots |c_n|^2) = \gamma\langle x, x\rangle = \gamma\|x\|^2.$$

So $\dfrac{\|Bx\|}{\|x\|} \leq \sqrt{\gamma}$. So $\|B\| \leq \sqrt{\gamma}$.

So $\|B\| = \gamma$ where $\gamma = \max\left\{\begin{matrix}\text{eigenvalues of} \\ B^*B\end{matrix}\right\}$.