Let (X,d) be a metric space. Let $A \subseteq X$.

A <u>sequence</u> in A is a function $\mathbb{Z}_{>0} \to A$
$n \mapsto x_n$.

Let $(a_1, a_2, \ldots)$ be a sequence in A.

A <u>limit point</u> of $(a_1, a_2, \ldots)$ is $z \in X$ such that
if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that
if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < \varepsilon$.
Write $\lim_{n\to\infty} a_n = z$, if z is a limit point of $(a_1, a_2, \ldots)$

A <u>cluster point</u> of $(a_1, a_2, \ldots)$ is $z \in X$ such that
there exists a subsequence $(a_{n_1}, a_{n_2}, \ldots)$
such that $\lim_{k\to\infty} a_{n_k} = z$.

<u>Exam question 1</u>: Show that, in metric spaces, limits of sequences are unique (when they exist).

Assume (X,d) is a metric space and $(x_1, x_2, \ldots)$ is a sequence in X. Let $z_1, z_2 \in X$ be limit points of $(x_1, x_2, \ldots)$. Then $z_1 = z_2$.

<u>Proof</u> Assume $\lim_{n\to\infty} x_n = z_1$ and $\lim_{n\to\infty} x_n = z_2$

To show: $z_1 = z_2$.

To show: $d(z_1, z_2) = 0$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $d(z_1, z_2) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: $d(z_1, z_2) < \varepsilon$.

Let $l_1 \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq l_1}$ then $d(a_n, z_1) < \frac{\varepsilon}{2}$

Let $l_2 \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq l_2}$ then $d(x_m, z_2) < \varepsilon/2$.

Then let $N = \max\{l_1, l_2\}$

$$d(z_1, z_2) \leq d(z_1, a_N) + d(a_N, z_2)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

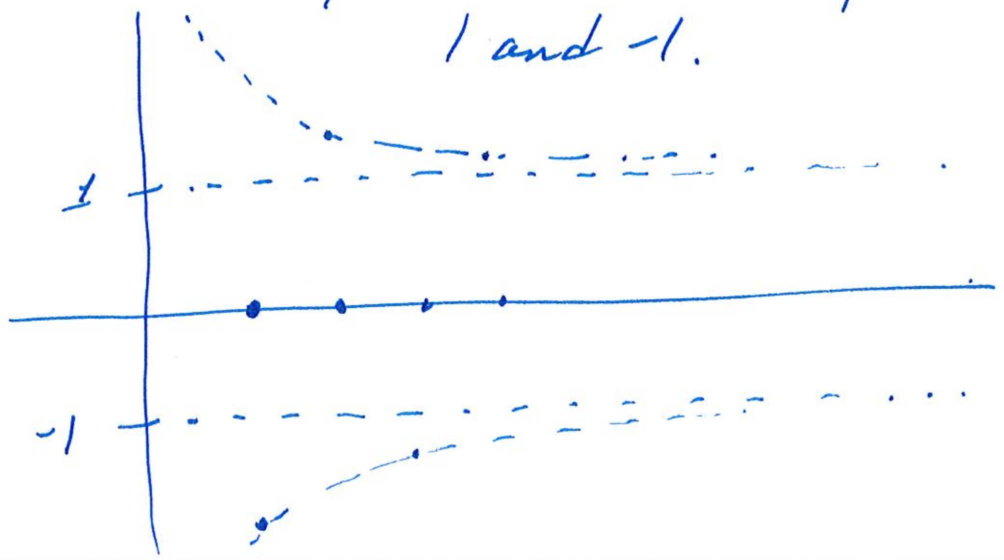
So $d(z_1, z_2) < \varepsilon$.

So $d(z_1, z_2) = 0$.

So $z_1 = z_2$ //.

Example $\begin{matrix} \mathbb{Z}_{>0} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & (-1)^n(1+\frac{1}{n}) \end{matrix}$ is the sequence

with no limit point and cluster points at 1 and -1.



Let (X,d) be a metric space. Let $A \subseteq X$.

A <u>Cauchy sequence</u> in A is a sequence $(a_1, a_2, \ldots)$ in A such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m,n \in \mathbb{Z}_{\geq \ell}$ then $d(a_m, a_n) < \varepsilon$.

<u>Exam question 2</u> Show that every convergent sequence is Cauchy.

Assume (X,d) is a metric space, $(x_1, x_2, \ldots)$ is a sequence in X and $z \in X$ and $\lim_{n\to\infty} x_n = z$.

To show: $(x_1, x_2 \ldots)$ is a Cauchy sequence.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $m,n \in \mathbb{Z}_{>0}$ then $d(x_m, x_n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $m,n \in \mathbb{Z}_{>0}$ then $d(x_m, x_n) < \varepsilon$.

Let $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, z) < \frac{\varepsilon}{2}$

Let $\ell = N$.

To show: If $m,n \in \mathbb{Z}_{\geq \ell}$ then $d(x_m, x_n) < \varepsilon$.

Assume $m, n \in \mathbb{Z}_{\geq \ell}$.

To show: $d(x_m, x_n) < \varepsilon$.

$$d(x_m, x_n) \leq d(x_m, z) + d(z, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $d(x_m, x_n) < \varepsilon$.

So $(x_1, x_2, \ldots)$ is a Cauchy sequence //.

Let (X, d) be a metric space. Let $A \subseteq X$.

The set A is <u>sequentially compact</u> if A satisfies:

if $(a_1, a_2, \ldots)$ is a sequence in A then there exists $z \in A$ such that z is a cluster point of $(a_1, a_2, \ldots)$.

The set A is <u>Cauchy compact</u> if A satisfies:

if $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then there exists $z \in A$ such that z is a limit point of $(a_1, a_2, \ldots)$.

The set A is <u>closed</u> if A satisfies:

if $(a_1, a_2, \ldots)$ is a sequence in A and $z \in X$ is a limit point of $(a_1, a_2, \ldots)$ then $z \in A$.