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Metric and Hilbert Spaces lect 6

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Let X and Y be sets.

A <u>function from X to Y</u> is a set $\Gamma_f \subseteq X \times Y$ such that

if $x \in X$ then there exists a unique $y \in Y$ such that $(x,y) \in \Gamma_f$.

Write
$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \quad \text{and} \quad \begin{matrix} f: X \to Y \\ x \mapsto f(x). \end{matrix}$$

Let S be a set.

A <u>relation on S</u> is a subset $R_\sim \subseteq S \times S$.

Write $a \sim b$ if $(a,b) \in R_\sim$

A <u>partially ordered set</u>, or <u>poset</u>, is a set S with a relation $\le$ ~~on~~ on S such that

(a) If $x \in S$ then $x \le x$,

(b) If $x,y,z \in S$ ~~then~~ and $x \le y$ and $y \le z$ then $x \le z$,

(c) If $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.

Let $(S, \le)$ be a poset. Let $E \subseteq S$.

An upper bound of E in S is $b \in S$ such that if $z \in E$ then $z \le b$.

A lower bound of E in S is $\ell \in S$ such that

if $z \in E$ then $\ell \le z$.

A least upper bound of E in S, or supremum,

is $\sup(E) \in S$ such that

(a) $\sup(E)$ is an upper bound of E in S

(b) If $b \in S$ is an upper bound of E in S then $\sup(E) \le b$.

A greatest lower bound, or infimum, of E in S

is $\inf(E) \in S$ such that

(a) $\inf(E)$ is a lower bound of E in S

(b) If ℓ is a lower bound of E in S then $\inf(E) \ge \ell$.

Proposition Let $(S, \le)$ be a poset. Let $E \subseteq S$. If $\sup(E)$ exists then $\sup(E)$ is unique.

Proof Let ~~$b_1 = \sup E$ and $b_2 = \sup E$~~.
b_1 and b_2 both be least upper bounds of E.
Then, since b_2 is an upper bound of E,

$$b_1 \le b_2.$$

Since b_1 is an upper bound of E then $b_2 \le b_1$.
So $b_1 = b_2$ //.

Interiors and Closures

Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.

The interior of A in X is $A^\circ \subseteq X$ such that

(a) A° is open in X and $A^\circ \subseteq A$,

(b) If U is open in X and $U \subseteq A$ then $U \subseteq A^\circ$.

The closure of A in X is $\bar{A} \subseteq X$ such that

(a) $\bar{A}$ is closed in X and $\bar{A} \supseteq A$,

(b) If C is closed in X and $C \supseteq A$ then $C \supseteq \bar{A}$.

A closed set in X is $C \subseteq X$ such that
C^c is open.

Recall: $C^c = \{x \in X \mid x \notin C\}$.

An <u>interior point of A</u> is $x \in X$ such that

~~∃~~ there exists $N \in \mathcal{N}(x)$ with $N \subseteq A$.

A <u>close point to A</u> is $x \in X$ such that

if $N \in \mathcal{N}(x)$ then $N \cap A \neq \emptyset$.

<u>Proposition</u> Let $(X, \mathcal{T})$ be a topological space.

Let $A \subseteq X$.

(a) $A^\circ = \{\text{interior points of } A\}$

(b) $\overline{A} = \{\text{close points of } A\}$.

<u>Proof of (a)</u>

Let $I = \{\text{interior points of } A\}$

$= \{x \in X \mid x \text{ is an interior point of } A\}$.

To show: $A^\circ = I$

To show: (aa) $A^\circ \subseteq I$

(ab) $A^\circ \supseteq I$.

(aa) To show: If $y \in A^\circ$ then $y \in I$.

Assume $y \in A^\circ$.

Then A° is open and $A^\circ \subseteq A$ and $y \in A^\circ$.

So y is an interior point of A

So $y \in I$.

So $A^\circ \subseteq I$.

(ab) To show: $A^\circ \supseteq I$.

To show: If $y \in I$ then $y \in A^\circ$.

Assume $y \in I$.

Then y is an interior point of A.

So there exists $N \in \mathcal{N}(y)$ with $N \subseteq A$.

So there exists $U \in \mathcal{T}$ with $y \in U \subseteq N \subseteq A$.

So $U \subseteq A^\circ$ (since A° is the largest open set contained in A).

So $y \in A^\circ$.

So $I \subseteq A^\circ$

So $I = A^\circ$. //