Alg. Geometry Week 10 Friday 05.10.2018
UniMelb
A. Ram

Examples of cohomology rings

Let X be a nice space (usually a smooth irreducible projective variety).

The Poincaré polynomial of X is

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \dim(H^j(X, \mathbb{Z}))\, q^j.$$

Examples

(pt) The point pt or the disc D^n or affine space $\mathbb{A}^n$

The Poincaré polynomial is 1, i.e.

$$H^*(pt, \mathbb{Z}) = H^*(D^n, \mathbb{Z}) = H^*(\mathbb{A}^n, \mathbb{Z}) = \text{span}\{1\}$$

with $\deg(1) = 0$ and $1^2 = 1$.

(S^n) The sphere S^n (Harder equation (4.38))

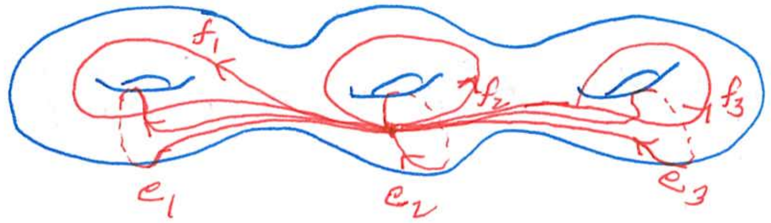
The Poincaré polynomial is $1 + q^n$, i.e.

$$H^*(S^n, \mathbb{Z}) = \frac{\mathbb{Z}[\omega]}{\langle \omega^2 = 0 \rangle} = \mathbb{Z}\text{-span}\{1, \omega\},$$

with $\deg(1) = 0$, $\deg(\omega) = n$ and $\omega^2 = 0$.

(Σ_g) Riemann surface Σ_g of genus g

/ Harder Exercise 22 p 77



The Poincaré polynomial is $1 + 2g\,q + q^2$,

$$H^*(\Sigma_g, \mathbb{Z}) = \mathrm{span}\{1\} \oplus \mathrm{span}\{f_g, \ldots, f_1, e_1, \ldots, e_g\} \oplus \mathrm{span}\{\omega\}$$

with $\deg(1) = 0$, $\deg(e_i) = \deg(f_i) = 1$, $\deg(\omega) = 2$

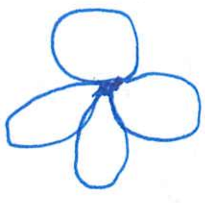
and products given by the multiplication table

$$\begin{array}{c|ccc|ccc} & f_g & \cdots & f_1 & e_1 & \cdots & e_g \\ \hline f_g & & & & & & \omega \\ \vdots & & 0 & & & \cdot^{\cdot^{\cdot}} & \\ f_1 & & & & \omega & & \\ \hline e_1 & & & -\omega & & & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & & 0 & \\ e_g & -\omega & & & & & \end{array}$$

and $\omega^2 = 0$.

$(\mathbb{C}^g/\Lambda)$ Complex tori and abelian varieties

$$\mathbb{C}^g/\Lambda \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong (\mathbb{R}/\mathbb{Z})^{2g} \cong (S^1)^{2g}$$

$S_1{}^4 =$ 

(Harder Equation (4.93) on p116 see also §4.8.10)

The Poincaré polynomial is $(1+q)^{2g}$,

$$(1+q)^{2g} = 1 + 2gq + \binom{2g}{2} q^2 + \cdots + \binom{2g}{2g-2} q^{2g-2} + 2g\, q^{2g-1} + q^{2g}.$$

$$H^*(\mathbb{C}^g/\Lambda, \mathbb{Z}) = H^*((S^1)^{2g}, \mathbb{Z}) = \Lambda^\cdot(f_g, \ldots, f_1, e_1, \ldots, e_g)$$

with $\deg(e_i) = \deg(f_i) = 1$.

($\Lambda^\cdot(f_g, \ldots, f_1, e_1, \ldots, e_g)$ is the exterior algebra generated by $e_1, \ldots, e_g, f_1, \ldots, f_g$.)

($\mathbb{P}^{n-1}$) <u>Projective space</u> $\mathbb{P}^{n-1}$ (Harder Exercise 26 page 107 and equation (4.118) on p147)

The Poincaré polynomial is

$$[n] = 1 + q^2 + q^4 + \cdots + q^{2(n-1)} = \frac{1-q^{2n}}{1-q^2}$$

$$H^*(\mathbb{P}^{n-1}; \mathbb{Z}) = \frac{\mathbb{Z}[\omega]}{\langle \omega^n = 0 \rangle} = \mathbb{Z}\text{-span}\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$$

with $\omega^n = 0$ and $\deg(\omega) = 2$.

Note that

$$\mathbb{P}^1 \cong S^2 \cong \Sigma_0$$

and the elliptic curves have $E_\tau \cong \mathbb{C}/\Lambda \cong (S^1)^2$.

$Gr_k(\mathbb{C}^n)$ Grassmannians $Gr_k(\mathbb{C}^n) = \{0 \subseteq V \subseteq \mathbb{C}^n \mid \dim_{\mathbb{C}} V = k\}$

The Poincaré polynomial is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-t^n)(1-t^{n-1})\cdots(1-t^{n-k+1})}{(1-t)(1-t^2)\cdots(1-t^k)} \quad \text{with } t = q^2.$$

(Fl) Flag varieties

$$Fl = \left\{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n \;\middle|\; \begin{array}{l} V_j \text{ a } \mathbb{C}\text{-submodule of } \mathbb{C}^n \\ \dim(V_j) = j \end{array} \right\}$$

The Poincaré polynomial is

$$[n]! = \frac{(1-t^n)}{(1-t)} \frac{(1-t^{n-1})}{(1-t)} \cdots \frac{(1-t)}{(1-t)} \quad \text{with } t = q^2.$$

(Fl_μ) Partial flag varieties $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$

$$Fl_\mu = \left\{ 0 \subseteq V_{\mu_1} \subseteq V_{\mu_1+\mu_2} \subseteq \cdots \subseteq V_{\mu_1+\cdots+\mu_\ell} = \mathbb{C}^n \;\middle|\; \begin{array}{l} V_j \text{ is a } \mathbb{C}\text{-submodule of } \mathbb{C}^n \\ \dim(V_j) = j \end{array} \right\}$$

The Poincaré polynomial is

$$\begin{bmatrix} n \\ \mu \end{bmatrix} = \frac{[n]!}{[\mu_1]!\,[\mu_2]!\cdots[\mu_\ell]!} \quad \text{with } t = q^2.$$

Sketch of proof Recall that

$$\mathbb{P}^{n-1} = Fl_{1,n-1}, \quad Fl = Fl_{1,1,\ldots,1} \quad \text{and} \quad Gr_k(\mathbb{C}^n) = Fl_{k,n-k}.$$

Alg. Geom. Week 10 Friday 05.10.2018 Uni Melb A. Ram (5)

The Bruhat decomposition

$$Fl_\mu = G/P_\mu = \bigsqcup_{w \in S_n/S_\mu} BwP_\mu,$$

where $S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_\ell}$ and the union is over coset representatives of the cosets of S_μ in S_n. If w is minimal length in the coset wS_μ and $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ is a reduced word then

$$BwP_\mu = \{ y_{i_1}(c_1) \cdots y_{i_{\ell(w)}}(c_{\ell(w)}) P_\mu \mid c_1, \ldots, c_{\ell(w)} \in \mathbb{C} \} \cong \mathbb{C}^{\ell(w)}$$

Then the Poincaré polynomial for Fl_μ is

$$\sum_{w \in S_n/S_\mu} q^{2\ell(w)} \qquad \left(\text{coming from } Fl_\mu = \bigsqcup_{w \in S_n/S_\mu} BwP_\mu\right)$$

It remains to verify that

$$\sum_{w \in S_n/S_\mu} q^{2\ell(w)} = \begin{bmatrix} n \\ \mu \end{bmatrix} = \frac{[n]!}{[\mu_1]! \cdots [\mu_\ell]!}$$

which is done by first showing $\sum_{w \in S_n} q^{\ell(w)} = [n]!$

Descriptions of the rings $H^*(Fl_\mu, \mathbb{Z})$ and $K(Fl_\mu)$

Let S_n act on $x_1, \ldots, x_n$ by permutations and extend this to an action on $\mathbb{Z}[x_1, \ldots, x_n]$,

$$wx_i = x_{w(i)}, \quad w(a_1f_1 + a_2f_2) = a_1(wf_1) + a_2(wf_2)$$

$$\text{and} \quad w(f_1f_2) = (wf_1)(wf_2),$$

for $a_1, a_2 \in \mathbb{Z}$ and $f_1, f_2 \in \mathbb{Z}[x_1, \ldots x_n]$

For a subgroup $K \subseteq S_n$ let

$$\mathbb{Z}[x_1, \ldots, x_n]^K = \left\{ f \in \mathbb{Z}[x_1, \ldots x_n] \ \middle|\ \begin{matrix}\text{if } w \in K \text{ then} \\ wf = f\end{matrix} \right\}$$

Then

$$H^*(Fl_\mu, \mathbb{Z}) = \frac{\mathbb{Z}[x_1, \ldots, x_n]^{S_\mu}}{\langle f(x_1, \ldots, x_n) - f(0, \ldots 0) \mid f \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \rangle}$$

Similarly,

$$K(Fl_\mu) = \frac{\mathbb{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_\mu}}{\langle f(X_1, \ldots, X_n) - f(1, \ldots, 1) \mid f \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n} \rangle}.$$

The Chern character is given by

$$\begin{aligned} K(Fl_\mu) &\xrightarrow{ch} H^*(Fl_\mu) \\ X_i &\longmapsto 1 + x_i + \frac{x_i^2}{2!} + \cdots \end{aligned}$$