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Radical ideals

Let A be a commutative $\mathbb{Z}$-algebra.
Let $f \in A$. The element f is nilpotent if there exists $n \in \mathbb{Z}_{>0}$ such that $f^n = 0$.

Let p be an ideal of A. The radical of p is

$$\sqrt{p} = \left\{ f \in A \;\middle|\; \begin{array}{l} \text{there exists } n \in \mathbb{Z}_{>0} \\ \text{such that } f^n = 0 \text{ in } A/p \end{array} \right\}$$

A radical ideal is an ideal p such that
$$\sqrt{p} = p.$$

HW: Show that if p is a prime ideal then p is a radical ideal.

HW: Show that if p is an ideal then $\sqrt{\sqrt{p}} = \sqrt{p}$.

Let $A = \mathbb{F}[x_1, \ldots, x_n]$ and $S \subseteq A$. Let

$$V(S) = \left\{ (c_1, \ldots, c_n) \in \mathbb{F}^n \;\middle|\; \begin{array}{l} \text{if } f \in S \text{ then} \\ f(c_1, \ldots, c_n) = 0 \end{array} \right\}$$

HW: Show that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ is the ideal generated by S.

HW: Show that, if p is an ideal in $A = \mathbb{F}[x_1, \ldots, x_n]$ then $V(p) = V(\sqrt{p})$.

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<u>Theorem</u> (Hilbert's Nullstellensatz)
Let $\mathbb{F}$ be a field.

$$\left\{\begin{matrix}\text{affine varieties}\\ \text{in } \mathbb{F}^n\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{radical ideals}\\ \text{in } \mathbb{F}[x_1,\dots x_n]\end{matrix}\right\}$$

$$\cup\!| \qquad\qquad\qquad \cup\!|$$

$$\left\{\begin{matrix}\text{irreducible}\\ \text{affine varieties}\\ \text{in } \mathbb{F}^n\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{prime ideals}\\ \text{in } \mathbb{F}[x_1,\dots,x_n]\end{matrix}\right\}$$

$$V(p) \longleftarrow\!\shortmid \quad p$$

Define

$$\mathbb{F}[x_1,\dots,x_n]_d = \mathbb{F}\text{span}\{x_{i_1}\cdots x_{i_d} \mid i_1,\dots,i_d\in\{1,\dots,n\}\}$$

$$= \left\{ f\in\mathbb{F}[x_1,\dots,x_n] \;\middle|\; \begin{matrix}\text{If } c\in\mathbb{C}^\times \text{ then}\\ f(cx_1,\dots,cx_n) = c^d f(x_1,\dots,x_n)\end{matrix}\right\}$$

so that

$$\mathbb{F}[x_1,\dots,x_n] = \bigoplus_{d\in\mathbb{Z}_{>0}} \mathbb{F}[x_1,\dots,x_n]_d$$

A <u>homogeneous ideal</u> is an ideal p of $\mathbb{F}[x_1,\dots,x_n]$ such that

$$p = \bigoplus_{d\in\mathbb{Z}_{\geq 0}} p_d, \quad \text{where } p_d = p\cap\mathbb{F}[x_1,\dots,x_n]_d.$$

Let $S \subseteq \mathbb{F}[x_1,\ldots,x_n]$ be a collection of homogeneous polynomials. Define

$$V_{\mathbb{P}}(S) = \left\{ [c_1,\ldots,c_n] \in \mathbb{P}^{n-1} \;\middle|\; \begin{array}{l} \text{if } f \in S \text{ then} \\ f(c_1,\ldots,c_n) = 0 \end{array} \right\}$$

<u>Theorem</u> (projective Nullstellensatz).

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{closed sets}\\ \text{in } \mathbb{P}^n\end{array}\right\} & \xleftrightarrow{\sim} & \left\{\begin{array}{c}\text{homogeneous}\\ \text{radical ideals}\\ \text{in } \mathbb{F}[x_1,\ldots,x_n]\end{array}\right\} \\
\cup\!| & & \cup\!| \\
\left\{\begin{array}{c}\text{irreducible}\\ \text{closed sets}\\ \text{in } \mathbb{P}^n\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c}\text{homogeneous}\\ \text{prime ideals}\\ \text{in } \mathbb{F}[x_1,\ldots,x_n]\end{array}\right\} \\
V_{\mathbb{P}}(\mathfrak{p}) & \longleftarrow & \mathfrak{p}
\end{array}$$

<u>HW</u>: Let $X \subseteq \mathbb{F}^n$ be a closed subset of $(\mathbb{F}^n, \mathcal{T}^{Zar}_{\mathbb{F}^n})$ and let $\mathfrak{p}$ be a radical ideal in $\mathbb{F}[x_1,\ldots,x_n]$ such that $X = V_{\mathbb{F}}(\mathfrak{p})$. Show that

$$H^0(X, \mathcal{O}_X) = \frac{\mathbb{F}[x_1,\ldots,x_n]}{\mathfrak{p}}.$$

<u>HW</u> Let X be an irreducible closed subset of $(\mathbb{P}^n, \mathcal{T}^{Zar}_{\mathbb{P}^n})$. Show that

$$H^0(X, \mathcal{O}_X) = \mathbb{F}.$$

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Irreducibility and dimension

Let $(X, \mathcal{T}_X)$ be a topological space.

The topological space $(X, \mathcal{T}_X)$ is <u>irreducible</u> if there do not exist $X_1, X_2 \subseteq X$ such that

X_1 is closed, X_2 is closed, $X_1 \neq \emptyset$, $X_2 \neq \emptyset$

$X_1 \neq X$, $X_2 \neq X$ and $X = X_1 \cup X_2$.

<u>HW</u>: Let $X = \{(c_1, c_2) \in \mathbb{F}^2 \mid c_1 c_2 = 0\}$ as a subspace of $(\mathbb{F}^2, \mathcal{T}^{Zar}_{\mathbb{F}^2}, \mathcal{O}_{\mathbb{F}^2})$. Show that X is reducible and determine its irreducible components.

Let $(X, \mathcal{T}_X)$ be a topological space.
The <u>dimension of X</u> is

$$\dim(X) = \sup\left\{ n \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \text{there exist closed irreducible subsets} \\ \emptyset \neq X_0 \subsetneq X \subsetneq \cdots \subsetneq X_n = X \end{array} \right\}$$

Let A be an integral $\mathbb{F}$-algebra.
The <u>dimension</u> of A is (Harder Definition 7.1.13)

$$\dim(A) = \sup\left\{ n \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{l} \text{there exist prime ideals} \\ 0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subsetneq X \end{array} \right\}$$

<u>HW</u>: Show that if $X = (\mathbb{F}^n, \mathcal{T}^{Zar}_{\mathbb{F}^n})$ then $\dim X = n$.

<u>HW</u>: Show that if $X = (\mathbb{P}^{n-1}, \mathcal{T}^{Zar}_{\mathbb{P}^{n-1}})$ then $\dim X = n-1$.

<u>HW</u>: Let X be an irreducible closed subset of $(\mathbb{F}^n, \mathcal{T}^{Zar}_{\mathbb{F}^n})$. Show that $\dim(X) = n-1$
if and only if
there exists $f \in \mathbb{F}[x_1, \ldots, x_n]$ with $f \notin \mathbb{F}$ and
f irreducible such that $X = V_{\mathbb{F}}(\{f\})$.

<u>HW</u>: Let X be an irreducible closed subset of $(\mathbb{P}^n, \mathcal{T}^{Zar}_{\mathbb{P}^n})$. Show that $\dim(X) = n-1$
if and only if
there exists $f \in \mathbb{F}[x_1, \ldots, x_n]$ with
f homogeneous, $f \notin \mathbb{F}$ and f irreducible
such that
$$X = V_{\mathbb{P}}(\{f\}).$$

The category of elliptic curves

An <u>elliptic curve</u> is a pair $(E, 0)$ with

$E \subseteq \mathbb{P}^n$ and $0 \in E$ such that

(a) E is closed in $(\mathbb{P}^n, \mathcal{T}^{Zar}_{\mathbb{P}^n})$

(b) E is irreducible

(c) $\dim(E) = 1$

(d) the arithmetic genus of E is 1.

Let $(E_1, 0_1)$ and $(E_2, 0_2)$ be elliptic curves.

An <u>isogeny</u> from $(E_1, 0_1)$ to $(E_2, 0_2)$ is a

morphism $\varphi : E_1 \to E_2$ such that $\varphi(0_1) = 0_2$.

The <u>category of elliptic curves</u> has (Harder Chapt. 10 p. 312 paragraph 2)

<u>Objects</u>: Elliptic curves $(E, 0)$

<u>Morphisms</u>: isogenies $E_1 \xrightarrow{\varphi} E_2$.

<u>HW</u>: Show that $\mathrm{Hom}(E_1, E_2)$ is a $\mathbb{Z}$-module of rank ≤ 4.

<u>HW</u>: Show that an isogeny is a group homomorphism.

<u>HW</u> Show that if (E, O) is an elliptic curve then there exist morphisms of projective varieties

$$\begin{matrix} E\times E \to E \\ (P,Q)\mapsto P+Q \end{matrix} \quad \text{and} \quad \begin{matrix} E\to E \\ P\mapsto -P \end{matrix}$$

(Harder Chapt. 5 page 211 sentence 3)

such that E becomes an abelian group with identity O.

Products

Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}\,\mathcal{C}$.

A <u>product</u> of X and Y is $X\times Y \in \mathrm{Ob}\,\mathcal{C}$ with morphisms

$$\pi_X : X\times Y \to X \quad \text{and} \quad \pi_Y : X\times Y \to Y$$

such that

if $Z \in \mathrm{Ob}\,\mathcal{C}$ and $f_X : X \to Z$ and $f_Y : Y \to Z$

then there exists a unique morphism $f : Z \to X\times Y$ such that $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$

$$\begin{tikzcd} & & Y \\ Z \arrow[r, dashed, "f"] \arrow[urr, bend left, "f_Y"] \arrow[drr, bend right, "f_X"'] & X\times Y \arrow[ur, "\pi_Y"] \arrow[dr, "\pi_X"'] & \\ & & X \end{tikzcd}$$

(Harder §1.3.3 Example 8 and Theorem 6.2.5)

<u>HW</u>: Show that if X and Y are irreducible closed sets in $(\mathbb{P}^n, \mathcal{T}^{Zar}_{\mathbb{P}^n})$ then $X\times Y$ is irreducible and closed in $(\mathbb{P}^n, \mathcal{T}^{Zar}_{\mathbb{P}^n})$.

Weierstrass equations

A **Weierstrass equation** is

$$y^2 = x^3 + Ax + B \quad \text{with } A, B \in \mathbb{F}.$$

(Harder eqn (5,53))

The **homogenization of the Weierstrass equation** is

$$y^2z - x^3 - Axz^2 - Bz^3 = 0.$$

Let

$$f = y^2 - x^3 - Ax - B, \quad \text{and}$$

$$f^* = y^2z - x^3 - Axz^2 - Bz^3.$$

Let

$$V_{\mathbb{F}}(f) = \{(c_1, c_2) \in \mathbb{F}^2 \mid f(c_1, c_2) = 0\}$$

$$V_{\mathbb{P}}(f^*) = \{[c_1, c_2, c_3] \in \mathbb{P}^2 \mid f^*(c_1, c_2, c_3) = 0\}$$

Define

$$\begin{matrix} \varphi: V_{\mathbb{F}}(f) & \longrightarrow & V_{\mathbb{P}}(f^*) \\ (c_1, c_2) & \longmapsto & [c_1, c_2, 1] \end{matrix}$$

HW: Show that φ is injective and

$$V_{\mathbb{P}}(f^*) = \operatorname{im}\varphi \sqcup \{[0,1,0]\}$$

(so that "$V_{\mathbb{P}}(f^*)$ is $V_{\mathbb{F}}(f)$ with a point at infinity").

The **discriminant** of $y^2 = x^3 + Ax + B$ is

$$\Delta = -16(4A^3 + 27B).$$

<u>Theorem</u>

(a) If $\Delta \neq 0$ then $(V_{\mathbb{P}}(f^*), [0,1,0])$ is an elliptic curve.

(b) This construction produces all elliptic curves (up to isomorphism).

<u>Exercise</u>: Show that $V_{\mathbb{P}}(f^*)$ is smooth if and only if $\Delta \neq 0$.

<u>Proof of $\Rightarrow$</u>: Let

$$D = \left(\frac{\partial f^*}{\partial x}, \frac{\partial f^*}{\partial y}, \frac{\partial f^*}{\partial z}\right) = (-3x^2 - Az^2, 2yz, y^2 - Ax + 3Bz^2)$$

Then

$$D\Big|_{[x,y,z] = [0,1,0]} = (0, 0, 1) \text{ which has rank } 1.$$

So $V_{\mathbb{P}}(f^*)$ is smooth at $[0,1,0]$.

Next

$$D\Big|_{[x,y,z] = [x,y,1]} = (-3x^2 - A, 2y, y^2 - Ax + 3B).$$

This has rank 0 only if all components are 0.

Assume $-3x^2 - A = 0$, $2y = 0$, $y^2 - Ax + 3B = 0$. Then $2y = 0$ and $4y^2 - 4Ax + 12B = 0$.

So $4Ax = 12B$. Since $-3x^2 - A = 0$ then $4^2A^2(-3x^2 - A) = 0$.

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So $-3(4Ax)^2 - 16A^3 = 0$.

So $-3(12B)^2 - 16A^3 = 0$.

So $-16(27B^2 + A^3) = 0$.

So $\Delta = 0$. //

<u>Cartoons for the Weierstrass equation with $A, B \in \mathbb{R}$</u>

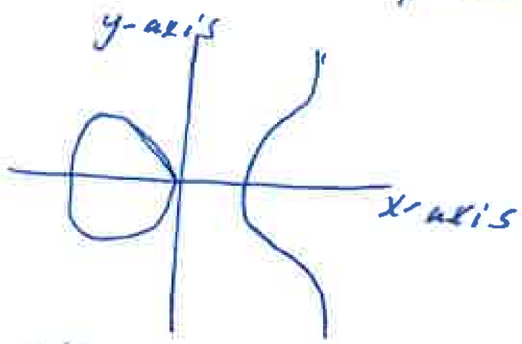
Let $V_{\mathbb{R}}(f) = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3 + Ax + B\}$

and $\Delta = -16(4A^3 + 27B^2)$.

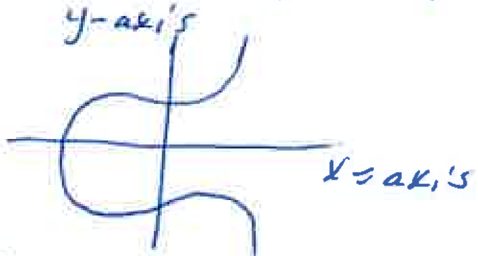
(a) Show that if $\Delta \neq 0$ then $V_{\mathbb{R}}(f)$ does <u>not</u> have

cusps or self intersections or isolated points.

(b) Show that if $A = -1$, $B = 0$ then $\Delta \in \mathbb{R}_{>0}$ and



(c) Show that if $A = -1$, $B = 1$ then $\Delta \in \mathbb{R}_{<0}$ and



(d) Show that $\Delta \in \mathbb{R}_{>0}$ if and only if $V_{\mathbb{R}}(f)$ has two connected components as a subset of $(\mathbb{R}^2, \mathcal{T}^{std}_{\mathbb{R}^2})$

(e) Show that $\Delta \in \mathbb{R}_{<0}$ if and only if $V_{\mathbb{R}}(f)$ has one connected componenent as a subset of $(\mathbb{R}^2, \mathcal{T}^{std}_{\mathbb{R}^2})$

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Arithmetic genus

Let X be an irreducible closed subset of $(\mathbb{P}^n, \mathcal{T}^{Zar}_{\mathbb{P}^n})$ and let p be a homogeneous ideal of $\mathbb{F}[x_1, \dots, x_n]$ such that $X = V_{\mathbb{P}}(\{p\})$.

Since

$$p = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} p_d \quad \text{with} \quad p_d = p \cap \mathbb{F}[x_1, \dots, x_n]_d$$

then

$$\frac{\mathbb{F}[x_1, \dots, x_n]}{p} = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \left(\frac{\mathbb{F}[x_1, \dots, x_n]}{p}\right)_d.$$

The Hilbert polynomial of X is $\mathcal{X}_X \in \mathbb{Q}[t]$ such that if $d \in \mathbb{Z}_{\geq 0}$ then $\mathcal{X}_X(d) = \dim\left(\left(\frac{\mathbb{F}[x_1, \dots, x_n]}{p}\right)_d\right)$

The constant term of $\mathcal{X}_X$ is $\mathcal{X}_X(0)$ and the arithmetic genus of X is

$$g = (-1)^{\dim X}(\mathcal{X}_X(0) - 1)$$

(Harder 5.1.1 and 58.3 page 157)

HW Show that if $\dim(X) = 1$ then $\mathcal{X}_X(t) = at + b$, with $a, b \in \mathbb{Q}$, and $g = -b + 1$.

HW: Show that if X is an elliptic curve then $\mathcal{X}_X(t) = 3t$ and $g = 1$

HW: Show that if $X = \mathbb{P}^n$ then

$$\mathcal{X}_{\mathbb{P}^n}(t) = \frac{1}{n!}(t+1)(t+2)\cdots(t+n).$$

Smoothness

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Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space and let $p \in X$.

The stalk of $\mathcal{O}_X$ at p is

$$\mathcal{O}_{X,p} = \varinjlim_{\substack{U \in \mathcal{T}_X \\ p \in U}} \mathcal{O}_X(U),$$

(Harder Definition 3.3.1)

where the set $\{U \in \mathcal{T}_X \mid p \in U\}$ is ordered by inclusion.

A locally ringed space is a ringed space $(X, \mathcal{T}_X, \mathcal{O}_X)$ such that

if $p \in X$ then $\mathcal{O}_{X,p}$ is a local ring.

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a locally ringed space. Let $p \in X$.
The space $(X, \mathcal{T}_X, \mathcal{O}_X)$ is smooth at p if

$$\dim_{k_p}(\mathfrak{m}_p/\mathfrak{m}_p^2) = \dim(\mathcal{O}_{X,p}),$$

(Harder Definition 7.5.1)

where $\mathfrak{m}_p$ is the maximal ideal of $\mathcal{O}_{X,p}$
and $k_p = \mathcal{O}_{X,p}/\mathfrak{m}_p$ is the residue field at p.

The space $(X, \mathcal{T}_X, \mathcal{O}_X)$ is smooth if $(X, \mathcal{T}_X, \mathcal{O}_X)$ satisfies

if $p \in X$ then $(X, \mathcal{T}_X, \mathcal{O}_X)$ is smooth at p.

Jacobi criterion for smoothness

(1) Let I be a radical ideal of $\mathbb{F}[x_1,\ldots,x_n]$ and let $X = V_{\mathbb{F}}(I)$. Let $f_1,\ldots,f_m$ be generators of I. Let $p \in X$. Then $(X, \mathcal{T}_X^{Zar}, \mathcal{O}_X)$ is smooth at p if and only if

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right) = n - \dim(X).$$

(Harder §7.5.3 Example 16 and Theorem 7.5.4)

$\left(\left(\frac{\partial f_i}{\partial x_j}\right) \text{ is an } m\times n \text{ matrix of polynomials}\right)$.

(2) Let I be a homogeneous radical ideal of $\mathbb{F}[x_1,\ldots,x_n]$ and let $X = V_{\mathbb{P}}(I)$. Let $f_1,\ldots,f_m \in \mathbb{F}[x_1,\ldots,x_n]$ be homogeneous generators of I. Let $p \in X$. Then $(X, \mathcal{T}_X, \mathcal{O}_X)$ is smooth at p if and only if

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(p)\right) = n - \dim X.$$

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Alg. Geom. Week 5 (14)

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Harder Definition 5.1.25

An elliptic curve is $\mathbb{C}/\Omega$, where Ω is a lattice in $\mathbb{C}$,

$$\Omega = \mathbb{Z}a + \mathbb{Z}b \quad \text{with } a, b \in \mathbb{C}.$$

An elliptic function is a meromorphic function on $\mathbb{C}/\Omega$.

The Weierstrass $\wp$ and $\wp'$ functions are

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

$$\wp'(z) = -2\sum_{\omega\in\Omega} \frac{1}{(z-\omega)^2}$$

Let

$$g_2(\Omega) = 60\sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \frac{1}{\omega^4} \quad \text{and} \quad g_3(\Omega) = 140\sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \frac{1}{\omega^6}$$

Then $\wp'(z)^2 = 4\wp(z)^3 - g_2(\Omega)\wp(z) - g_3(\Omega)$

Then (Harder Lemma 5.1.27)

$$\begin{array}{rcl} \mathbb{C}/\Omega & \xrightarrow{f} & \mathbb{P}^2 \\ z & \longmapsto & [\wp'(z), \wp(z), 1] \end{array} \quad \text{is injective}$$

and

$$X = \operatorname{im} f = \{[x,y,u] \in \mathbb{P}^2 \mid y^2u - 4x^3 - g_2(\Omega)xu^2 - g_3(\Omega)u^3 = 0\}$$

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Elliptic curve as a scheme (Harder Chapt 5 page 212 paragraph 1)

Let
$$E = \{ [x,y,z] \in \mathbb{P}^2 \mid y^2 z - 4x^3 - g_2(\Omega) x z^2 - g_3(\Omega) z^3 = 0 \}$$
with $0 = [0,1,0]$.

Let
$$U_0 = \{ [x,y,z] \in E \mid z \neq 0 \} = E - \{[0,1,0]\}$$
$$U_1 = \{ [x,y,z] \in E \mid y \neq 0 \}$$

Then
$$\mathcal{O}_E(U_0) = \mathbb{C}[x,y,1] \quad \text{and} \quad \mathcal{O}_E(U_1) = \mathbb{C}[\tfrac{x}{y}, 1, \tfrac{1}{y}]$$