The forgetful functor

$$\begin{aligned} Sh(X) &\longrightarrow PSh(X) \\ \mathcal{F} &\longmapsto \mathcal{F} \end{aligned}$$

The direct image functor

Let $f: X \to Y$ be a morphism. Define

$$f_*: Sh(X) \longrightarrow Sh(Y)$$

by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)), \quad \text{for } V \in \mathcal{T}_Y$$

Adjoints

Sheafification $\quad PSh(X) \longrightarrow Sh(X)$

$$\mathcal{F} \longmapsto \mathcal{F}^\#$$

determined by

$$Hom_{Sh(X)}(\mathcal{F}^\#, \mathcal{G}) = Hom_{PSh(X)}(\mathcal{F}, \mathcal{G})$$

for $\mathcal{F} \in PSh(X)$ and $\mathcal{G} \in Sh(X)$.

Inverse image Let $f: X \to Y$ be a morphism.

$$f^*: Sh(Y) \longrightarrow Sh(X)$$

determined by

$$Hom_{Sh(X)}(f^*\mathcal{G}, \mathcal{F}) = Hom_{Sh(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

The map $X \to pt$

$$\begin{array}{ccc} X & \xrightarrow{\pi} & pt \\ x & \longmapsto & * . \end{array}$$

Note that

$$Sh(pt) = \left\{\begin{matrix}\mathbb{Z}\text{-modules} \\ \text{on } pt\end{matrix}\right\} = \left\{\mathbb{Z}\text{-modules}\right\} = \left\{\begin{matrix}\text{abelian} \\ \text{groups}\end{matrix}\right\}$$

Global sections functor

$$H^0(X,-) : Sh(X) \to Sh(pt)$$

given by

$$H^0(X,\mathcal{F}) = \pi_* \mathcal{F}.$$

Constant sheaf functor

$$\begin{array}{ccc} Sh(pt) & \longrightarrow & Sh(X) \\ \mathbb{F} & \longmapsto & \underline{\mathbb{F}} \end{array}$$

given by

$$\underline{\mathbb{F}} = \pi^* \mathbb{F}$$

HW: Show that $H^0(X,\mathcal{F}) = \mathcal{F}(X)$.

HW: Show that

$$\underline{\mathbb{F}}(U) = \left\{ f : U \to \mathbb{F} \;\middle|\; \begin{matrix} f \text{ is continuous} \\ \text{for } (\mathbb{F}, \mathcal{T}_{\mathbb{F}}^{disc}) \end{matrix} \right\}$$

(see Harder §3.1.4 Example (4)).

<u>The map $pt \xrightarrow{\iota_a} X$</u>

Let $a \in X$ and let

$$\begin{array}{rcl} pt & \xrightarrow{\iota_a} & X \\ * & \longmapsto & a \end{array}$$

<u>Skyscraper sheaf functor</u>

$$\begin{array}{rcl} Sh(pt) & \longrightarrow & Sh(X) \\ \mathbb{F} & \longmapsto & \mathbb{F}^a \end{array}$$ (Harder p. 54 Vol. I)

is given by $\mathbb{F}^a = (\iota_a)_*(\mathbb{F})$.

<u>Stalk functor</u>

$$\begin{array}{rcl} Sh(X) & \longrightarrow & Sh(pt) \\ \mathcal{F} & \longmapsto & \mathcal{F}_a \end{array}$$

is given by $\mathcal{F}_a = \iota_a^*(\mathcal{F})$.

<u>HW</u> Show that

$$\mathcal{F}_a = \varinjlim_{\substack{U \in \mathcal{T}_X \\ a \in U}} \mathcal{F}(U)$$

<u>HW</u> Show that

$$\mathbb{F}^a(U) = \begin{cases} \mathbb{F}, & \text{if } a \in U, \\ 0, & \text{otherwise.} \end{cases}$$

<u>HW</u> Show that

$$(\mathbb{F}^a)_p = \begin{cases} \mathbb{F}, & \text{if } p = a, \\ 0, & \text{if } p \neq a. \end{cases}$$

Constructing f^* and $\mathcal{F}^\#$

HW Let $f: X \to Y$ be a morphism. Let $\mathcal{G} \in Sh(Y)$. Show that the inverse image $f^*\mathcal{G}$ exists in $Sh(X)$ and is given by

$$f^*\mathcal{G} = \tilde{\mathcal{G}}^\# \quad \text{where } \tilde{\mathcal{G}}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

HW Let $\mathcal{F} \in PSh(X)$.

For $y \in X$ let s_y be given by
$$\begin{array}{rcl} \mathcal{F}(V) & \to & \mathcal{F}_y = \varinjlim_{\substack{U \in \mathcal{T}_X \\ y \in U}} \mathcal{F}(U) \\ s & \mapsto & s_y. \end{array}$$

Show that the sheafification $\mathcal{F}^\#$ exists in $Sh(X)$ and is given by

$$\mathcal{F}^\#(U) = \left\{ \breve{s} = (\breve{s}_x) \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \text{if } x \in U \text{ then there exist} \\ V \in \mathcal{T}_X \text{ and } s \in \mathcal{F}(V) \text{ with} \\ V \subseteq U,\ x \in V \text{ and } (\breve{s}_y)_{y \in V} = (s_y)_{y \in V} \end{array} \right\}$$

(see Harder equation (3.11)).

HW Let $\mathcal{F} \in PSh(X)$ and let $\mathcal{F}_{\sqcup} = \bigsqcup_{p \in X} \mathcal{F}_p$

with $\mathcal{T}_{\mathcal{F}_\sqcup}$ minimal such that the maps

$$\begin{array}{rcl} \breve{s}: U & \to & \mathcal{F}_\sqcup, \\ p & \mapsto & s_p \end{array} \quad \text{for } s \in \mathcal{F}(U)$$

are continuous. Let $\begin{array}{c} \mathcal{F}_\sqcup \\ \pi \downarrow \\ X \end{array}$ be given by $\pi(s_x) = x$ for $s_x \in \mathcal{F}_x$. Show that

$$\mathcal{F}^\#(U) = \left\{ \breve{s}: U \to \mathcal{F}_\sqcup \;\middle|\; \begin{array}{l} \breve{s} \text{ is continuous and} \\ \pi \circ \breve{s} = id_U \end{array} \right\}$$