Algebraic Geometry Week 8 — 11/09/2018 — UniMelb — A. Ram — ①

Cohomologies

Sheaf cohomology (Harder equation (4.12))

$$H^\bullet(X,-): \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } pt\end{matrix}\right\}$$

is the right derived functor to

$$H^0(X,-): \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } pt\end{matrix}\right\}$$

Cohomology with compact supports (Harder §4.7.1)

$$H^\bullet_c(X,-): \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } pt\end{matrix}\right\}$$

is the right derived functor to

$$H^0(X,-): \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } pt\end{matrix}\right\}$$

Direct image (Harder equation (4.29)).

Let $f: X \to Y$ be a continuous map.

The direct image

$$R^\bullet f_*: \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } Y\end{matrix}\right\}$$

is the right derived functor to

$$f_*: \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } Y\end{matrix}\right\}$$

The universal property of cohomology

Direct image Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \text{ in } \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on } \mathcal{O}_X \end{matrix} \right\}$$

then there is an exact sequence in $\left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on } Y \end{matrix} \right\}$

$$0 \to f_*(\mathcal{F}') \to f_*(\mathcal{F}) \to f_*(\mathcal{F}'') \to$$
$$\to R^1f_*(\mathcal{F}') \to R^1f_*(\mathcal{F}) \to R^1f_*(\mathcal{F}'') \to$$
$$\to \cdots$$

Sheaf cohomology Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \text{ in } \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on } \mathcal{O}_X \end{matrix} \right\}$$

then there is an exact sequence in $\left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on pt} \end{matrix} \right\}$

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to$$
$$\to H^1(X, \mathcal{F}') \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{F}'') \to$$
$$\to \cdots$$

Cohomology with compact supports

Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \text{ in } \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on } X \end{matrix} \right\}$$

then there is an exact sequence in $\left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on pt} \end{matrix} \right\}$

$$0 \to H^0_c(X, \mathcal{F}') \to H^0_c(X, \mathcal{F}) \to H^0_c(X, \mathcal{F}'') \to$$
$$\to H^1_c(X, \mathcal{F}') \to H^1_c(X, \mathcal{F}) \to H^1_c(X, \mathcal{F}'') \to$$
$$\to \cdots$$

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Construction of cohomologies

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<u>Direct image</u> Let $f: X \to Y$ be a continuous map. Let $\mathcal{F} \in \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on } X \end{matrix} \right\}$. Let

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

be an injective resolution of $\mathcal{F}$. Then

$$R^q f_*(\mathcal{F}) = \frac{\ker(f_*(\mathcal{I}^q) \to f_*(\mathcal{I}^{q+1}))}{\operatorname{im}(f_*(\mathcal{I}^{q-1}) \to f_*(\mathcal{I}^q))}$$

<u>Sheaf cohomology</u> Let $f: X \to pt$. Then

$$H^q(X, \mathcal{F}) = R^q f_*(\mathcal{F}) = \frac{\ker(H^0(X, \mathcal{I}^q) \to H^0(X, \mathcal{I}^{q+1}))}{\operatorname{im}(H^0(X, \mathcal{I}^{q-1}) \to H^0(X, \mathcal{I}^q))}$$

<u>Cohomology with compact supports</u>

Let $\mathcal{F} \in \left\{ \begin{matrix} \mathbb{Z}\text{-modules} \\ \text{on } X \end{matrix} \right\}$. Let

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

be an injective resolution of $\mathcal{F}$. Then

$$H^q_c(X, \mathcal{F}) = \frac{\ker(H^0_c(X, \mathcal{I}^q) \to H^0_c(X, \mathcal{I}^{q+1}))}{\operatorname{im}(H^0_c(X, \mathcal{I}^{q-1}) \to H^0_c(X, \mathcal{I}^q))}$$

Injective resolutions

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space.

An <u>injective</u> <u>$\mathcal{O}_X$-module</u> is an $\mathcal{O}_X$-module $\mathcal{I}$ such that

if $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{I}$

are $\mathcal{O}_X$-module morphisms such that

$$\ker \varphi \subseteq \ker(\psi)$$

then there exists a unique $\mathcal{O}_X$-module morphism

$\eta : \mathcal{B} \to \mathcal{I}$ such that $\psi = \eta \circ \varphi$

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \mathcal{B} \arrow[dl, "\eta"] \\ \mathcal{I} & \end{tikzcd}$$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. An <u>injective resolution</u> of $\mathcal{F}$ is an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

with $\mathcal{I}^0, \mathcal{I}^1, \mathcal{I}^2, \ldots$ injective.

The source functors

Direct image Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a continuous map. The direct image

$$f_*: \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\mathbb{Z}\text{-modules}\\ \text{on } Y\end{matrix}\right\}$$

is given by

$$(f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V)), \quad \text{for } V \in \mathcal{T}_Y.$$

Global sections Let $\mathcal{F}$ be a $\mathbb{Z}$-module on X. Let $\pi: X \to pt$. Then

$$H^0(X, \mathcal{F}) = \pi_*(\mathcal{F}) = \mathcal{F}(X).$$

Global sections with compact supports.

Let $(X, \mathcal{T}_X)$ be a locally compact topological space. Let $\mathcal{F}$ be a $\mathbb{Z}$-module on X. Define

$$H^0_c(X, \mathcal{F}) = \left\{ s \in H^0(X, \mathcal{F}) \;\middle|\; \begin{matrix}\text{Supp}(s) \text{ is}\\ \text{compact}\end{matrix} \right\}$$

where

$$\text{Supp}(s) = \{ p \in X \mid s_p \neq 0 \}$$

where $s_p \in \mathcal{F}_p$ is a representative of s in

$$\mathcal{F}_p = \varinjlim_{\substack{U \in \mathcal{T}_X \\ p \in U}} \mathcal{F}(U),$$

the stalk of $\mathcal{F}$ at p.

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Čech cohomology (Harder §4.5)

Let $(X,\mathcal{T}_X)$ be a topological space and let $\mathcal{S}$ be an open cover of X.

Let F be a $\mathbb{Z}$-module on X. Define

$$0 \to C^0(X,\mathcal{S},F) \xrightarrow{d} C^1(X,\mathcal{S},F) \xrightarrow{d} C^2(X,\mathcal{S},F) \xrightarrow{d} \cdots$$

by

$$C^0(X,\mathcal{S},F) = \prod_{U\in\mathcal{S}} F(U),$$

$$C^1(X,\mathcal{S},F) = \prod_{U,V\in\mathcal{S}} F(U\cap V),$$

$$C^2(X,\mathcal{S},F) = \prod_{U,V,W\in\mathcal{S}} F(U\cap V\cap W), \quad \ldots$$

with

$$(dc)_{U\cap V} = c_V - c_U, \quad \text{for } c=(c_U)_{U\in\mathcal{S}},$$

$$(dc)_{U\cap V\cap W} = c_{V\cap W} - c_{U\cap W} + c_{U\cap V}, \quad \text{for } c=(c_{U\cap V})_{U,V\in\mathcal{S}}$$

$$(dc)_{U\cap V\cap W\cap Z} = c_{V\cap W\cap Z} - c_{U\cap W\cap Z} + c_{U\cap V\cap Z} - c_{U\cap V\cap W}, \ldots$$

Define

$$\check{H}(X,\mathcal{S},F) = \frac{\ker(C^q(X,\mathcal{S},F) \xrightarrow{d} C^{q+1}(X,\mathcal{S},F))}{\operatorname{im}(C^{q-1}(X,\mathcal{S},F) \xrightarrow{d} C^q(X,\mathcal{S},F))}$$

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Direct and Inverse images

Let

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \text{ be continuous.}$$

The direct image functor

$$f_* : Sh(X) \to Sh(Y)$$

is given by

$$(f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V)), \quad \text{for } V \in \mathcal{T}_Y$$

and $\mathcal{F} \in Sh(X)$.

The inverse image functor

$$f^{-1} : Sh(Y) \to Sh(X)$$

is the adjoint of f_*,

$$\mathrm{Hom}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{Sh(Y)}(\mathcal{G}, f_* \mathcal{F})$$

for $\mathcal{G} \in Sh(Y)$ and $\mathcal{F} \in Sh(X)$.

The inverse image $f^{-1}\mathcal{G}$ is constructed by

$$f^{-1}\mathcal{G} = \tilde{\mathcal{G}}^{\#},$$

where $\displaystyle \tilde{\mathcal{G}}(U) = \lim_{V \ni f(U)} \mathcal{G}(V)$ for $U \in \mathcal{T}_X$

and $\tilde{\mathcal{G}}^{\#}$ is the sheafification of $\tilde{\mathcal{G}}$.

Global sections and constant sheaf functors

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The global sections functor

$$H^0(X,-): Sh(X) \to Sh(pt) \quad \text{is}$$

$$H^0(X,\mathcal{F}) = f_*\mathcal{F} \quad \text{where } f: X \to pt.$$

The constant sheaf functor

$$\begin{array}{ccc} Sh(pt) & \longrightarrow & Sh(X) \\ \mathbb{F} & \longmapsto & \underline{\mathbb{F}} \end{array}$$

is given by

$$\underline{\mathbb{F}} = f^{-1}\mathbb{F} \quad \text{where } f: X \to pt.$$

Note that

$$Sh(pt) = \{\mathbb{Z}\text{-modules}\} = \{\text{abelian groups}\}$$

HW Show that if $\mathcal{F} \in Sh(X)$ then

$$H^0(X,\mathcal{F}) = \mathcal{F}(X)$$

HW Show that if $\mathbb{F} \in Sh(pt)$ then

$$\begin{aligned} \underline{\mathbb{F}}(U) &= \{\text{locally constant functions on } U\} \\ &= \left\{ f: U \to \mathbb{F} \;\middle|\; \begin{array}{l} f \text{ is continuous} \\ \text{for } (\mathbb{F}, \mathcal{T}_{\mathbb{F}}^{disc}) \end{array} \right\} \end{aligned}$$

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The functor $\mathrm{Hom}_R(X,-)$

Let $R = \mathcal{O}_X(X)$ be a commutative ring.

Let R-mod be the category of R-modules.

Goal: Understand the category R-mod.

Let $X \in R$-mod. Define a functor

$$\begin{array}{rcl} \mathrm{Hom}_R(X,-) : R\text{-mod} & \longrightarrow & R\text{-mod} \\ Y & \longmapsto & \mathrm{Hom}_R(X,Y) \\ f: Y \to Z & \longmapsto & f_*: \mathrm{Hom}_R(X,Y) \to \mathrm{Hom}_R(X,Z) \\ & & \varphi \longmapsto f\circ\varphi \end{array}$$

If

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{is exact}$$

in R-mod then

$$0 \to \mathrm{Hom}_R(X,A) \xrightarrow{f_*} \mathrm{Hom}_R(X,B) \xrightarrow{g_*} \mathrm{Hom}_R(X,C)$$

is exact, but g_* is not necessarily surjective.

Example Let $R = \mathbb{Z}$ and $X = \mathbb{Z}/2\mathbb{Z}$. In $\mathbb{Z}$-mod

$$\begin{array}{c} 0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0 \quad \text{is exact} \\ z \longmapsto 2z \end{array}$$

and

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) & \xrightarrow{f_*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ \| & \| & & \| & & \\ 0 \to & 0 & \xrightarrow{f_*} & 0 & \xrightarrow{g_*} & \{+1,-1\} \end{array}$$

with g_* not surjective.

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Projective and injective R-modules

Many useful functors fail to preserve short exact sequences $0 \to A \to B \to C \to 0$.

<u>Question</u>: Can we classify all $X \in R$-mod for which $\mathrm{Hom}_R(X, -)$ is exact (i.e. preserves short exact sequences).

<u>Answer/Proposition</u>: $\mathrm{Hom}_R(X, -)$ is exact if and only if X is a projective R-module.

A <u>projective R-module</u> is an R-module P such that

if $M \xrightarrow{\pi} X$ and $P \xrightarrow{g} X$ and $\mathrm{im}(\pi) = X$

then there exists $P \xrightarrow{h} M$ such that $\pi \circ h = g$

$$\begin{tikzcd} & M \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"] & X \end{tikzcd}$$

(Harder Definition 2.3.7)

An <u>injective R-module</u> is an R-module I such that

if $N \xrightarrow{z} M$ and $N \xrightarrow{g} I$ and $\ker(z) \subseteq \ker(g)$

then there exists $M \xrightarrow{h} I$ such that $h \circ z = g$

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \\ I & N \arrow[l, "g"] \arrow[u, "z"'] \end{tikzcd}$$

(Harder Definition 2.3.5)

The universal property of $\mathrm{Ext}^i_R(X,-)$

Question: Can we keep track of the failure of exactness if X is not projective?

Answer/Proposition: There exist functors

$$\mathrm{Ext}^i_R(X,-) : R\text{-mod} \to R\text{-mod} \quad \text{for } i \in \mathbb{Z}_{\geq 0}$$

such that

(a) $\mathrm{Ext}^0_R(X,-) = \mathrm{Hom}_R(X,-)$,

(b) $\mathrm{Ext}^i_R(X,-) = 0$ for $i \in \mathbb{Z}_{>0}$ if and only if X is projective.

(c) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact then there is an exact sequence

$$0 \to \mathrm{Hom}_R(X,A) \to \mathrm{Hom}_R(X,B) \to \mathrm{Hom}_R(X,C)$$
$$\to \mathrm{Ext}^1_R(X,A) \to \mathrm{Ext}^1_R(X,B) \to \mathrm{Ext}^1_R(X,C)$$
$$\to \mathrm{Ext}^2_R(X,A) \to \mathrm{Ext}^2_R(X,B) \to \mathrm{Ext}^2_R(X,C)$$
$$\to \cdots$$

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Construction of $Ext^i_R(X,-)$

Let $Y \in R$-mod.

Let

$$0 \to Y \to I^0 \to I^1 \to \cdots \qquad (I^\bullet)$$

be an injective resolution of Y (an exact sequence with $I^0, I^1, I^2, \ldots$ injective).

HW: Check that R-mod "has enough injectives", i.e. that $0 \to Y \to I^0 \to I^1 \to \cdots$ exists.

Apply $Hom_R(X,-)$ to $I^\bullet$:

$$0 \to Hom_R(X, I^0) \to Hom_R(X, I^1) \to \cdots$$

HW: Show that is sequence is a chain complex and that is not necessarily exact.

Define

$$Ext^i_R(X,Y) = \frac{\ker(Hom_R(X, I^i) \to Hom_R(X, I^{i+1}))}{im(Hom_R(X, I^{i-1}) \to Hom_R(X, I^i))}$$

HW: Show that $Ext^i_R(X,Y)$ is independent (up to isomorphism) of the choice of $I^\bullet$.

Understanding Spec, Spec($\mathbb{C}[z,z^{-1}]$) 10.09.2018
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Use $\mathbb{C}[z] \hookrightarrow \mathbb{C}[z,z^{-1}]$ to understand $\mathbb{C}[z,z^{-1}]$.

If $R \xrightarrow{\varphi} S$ is a ring homomorphism and

$X = \mathrm{Spec}(S) = \{\text{prime ideals in } S\}$

$Y = \mathrm{Spec}(R) = \{\text{prime ideals in } R\}$

and $\mathrm{Spec}(\varphi): X \longrightarrow Y$
$p \longmapsto \varphi^{-1}(p)$.

If $r_1, r_2 \in R$ and $r_1 r_2 \in \varphi^{-1}(p)$ then $\varphi(r_1 r_2) \in p$.
So $\varphi(r_1)\varphi(r_2) \in p$ giving $\varphi(r_1) \in p$ or $\varphi(r_2) \in p$.
So
$r_1 \in \varphi^{-1}(p)$ or $r_2 \in \varphi^{-1}(p)$.
So $\varphi^{-1}(p)$ is a prime ideal.

Let p be prime ideal in $\mathbb{C}[z,z^{-1}]$.
Let $f \in p$ with factorization $f = z^{-k} f_1$ with $f_1 \in \mathbb{C}[z]$.
Since z^{-k} is a unit in $\mathbb{C}[z,z^{-1}]$ then $z^{-k} \notin p$ and
so $f_1 \in p$. So a linear factor of f_1 is in p.
So $p = z - c$ for some $c \in \mathbb{C}$.

All the ideals $(z-c)\mathbb{C}[z,z^{-1}]$ are maximal in $\mathbb{C}[z,z^{-1}]$
$(z-c)\mathbb{C}[z,z^{-1}]$ for $c \in \mathbb{C}^\times$.
are maximal in $\mathbb{C}[z,z^{-1}]$ since $\dfrac{\mathbb{C}[z,z^{-1}]}{(z-c)\mathbb{C}[z,z^{-1}]} \cong \mathbb{C}$.

Understanding Spec: Spec($\mathbb{F}$) when $\mathbb{F}$ is a field

Let $\mathbb{F}$ be a field.

Then $\mathbb{F}$ has only two ideals 0 and $\mathbb{F}$, and 0 is the only prime ideal.

So
$$\mathrm{Spec}(\mathbb{F}) = (X, \mathcal{T}_X^{\mathrm{Zar}}, \mathcal{O}_X) \text{ with}$$
$$X = \{*\} \quad \text{where} \quad * = (0).$$

If $S \subseteq \mathbb{F}$ then
$$\begin{aligned}
V(S) &= \{p \in X \mid \text{If } g \in S \text{ then } g = 0 \text{ in } \mathbb{F}/p\} \\
&= \{p \in X \mid \text{If } g \in S \text{ then } g \in p\} \\
&= \{p \in X \mid S \subseteq p\} \\
&= \begin{cases} \{0\}, & \text{if } S \subseteq 0, \\ \phi, & \text{if } S \not\subseteq 0 \end{cases} \\
&= \begin{cases} \{*\}, & \text{if } S = \{0\} \text{ or } S = \phi \\ \phi, & \text{if } S \neq \{0\} \text{ and } S \neq \phi \end{cases}
\end{aligned}$$

So
$$\mathcal{T}_X^{\mathrm{Zar}} = \{\phi, X\}$$
which is, after all, the only topology on a one point space.

By definition
$$X_g = \{p \in X \mid g \notin p\} \quad \text{so that}$$

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$X_0 = \{p \in X \mid 0 \notin p\} = \emptyset$ and

$X_1 = \{p \in X \mid 1 \notin p\} = X$

Then the structure sheaf $\mathcal{O}_X$ is given by

$\mathcal{O}_X(X) = \mathcal{O}_X(X_1) = \mathbb{F}$, since $X = X_1$

$\mathcal{O}_X(\emptyset) = \mathcal{O}_X(X_0) = 1$, since $\emptyset = X_0$.