(1) Constraints $z^2 = x^2 + y^2$ and $z = 1 + x + y$.

Minimse/Maximise $f(x,y,z) = x^2 + y^2 + z^2$.

Solution with three variables and two constraints,

$$g_1(x,y,z) = x^2 + y^2 - z^2, \quad g_2 = 1 + x + y - z$$

$$f(x,y,z) = x^2 + y^2 + z^2.$$

Then $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ gives

$$(2x, 2y, 2z) = \lambda_1 (2x, 2y, -2z) + \lambda_2 (1, 1, -1)$$

So

$$2x = \lambda_1 2x + \lambda_2 \qquad \text{So } 2x - 2y = \lambda_1 2x + \lambda_2 - \lambda_1 2y - \lambda_2$$
$$2y = \lambda_1 2y + \lambda_2 \qquad = 2\lambda_1 (x - y)$$
$$2z = -\lambda_1 2z - \lambda_2. \qquad \text{So } x - y = \lambda_1 (x - y)$$

So $(1 - \lambda_1)(x - y) = 0$. So $\lambda_1 = 1$ or $x = y$.

<u>Case 1</u> $\lambda_1 = 1$.

Then
$$2x = 2x + \lambda_2 \qquad \text{So } 0 = \lambda_2 \qquad \text{So } z = 0$$
$$2y = 2y + \lambda_2 \qquad 4z = -\lambda_2$$
$$2z = -2z - \lambda_2$$

Then $z^2 = x^2 + y^2$ gives $x = 0$ and $y = 0$, which is a contradiction to $z = 1 + x + y$.

So this case cannot occur.

Case 2: $x = y$

Then the constraint $g_2(x,y,z) = 0$ gives

$0 = 1 + x + y - z = 1 + 2x - z$ so that $z = 1 + 2x$.

Then the constraint $g_1(x,y,z) = 0$ gives

$0 = x^2 + y^2 - z^2 = x^2 + \cancel{x}^2 - (1+2x)^2 = -1 - 4x - 2x^2$

and so $0 = x^2 + 2x + \frac{1}{2} = (x^2 + 2x + 1) - \frac{1}{2} = (x+1)^2 - \frac{1}{2}$

So $x + 1 = \pm\frac{1}{\sqrt{2}}$ and $x = -1 \pm \frac{1}{\sqrt{2}}$.

So the critical points are at $x = -1 \pm \frac{1}{\sqrt{2}}$, $x = y$, $z = 1 + x + y$.

$(x,y,z) = (-1 + \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}, -1 + \sqrt{2})$

$(x,y,z) = (-1 - \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}, -1 - \sqrt{2})$

For the first point $f(x,y,z) = 2(-1+\sqrt{2})^2 = 6 - 4\sqrt{2}$

For the second point $f(x,y,z) = 2(-1-\sqrt{2})^2 = 6 + 4\sqrt{2}$.

So $(x,y,z) = (-1 - \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}, -1 - \sqrt{2})$ is a point on C local minimum from the origin (distance $\sqrt{6 + 4\sqrt{2}}$)

and $(x,y,z) = (-1 + \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}, -1 + \sqrt{2})$ is the point on C closest to the origin (distance $\sqrt{6 - 4\sqrt{2}}$).

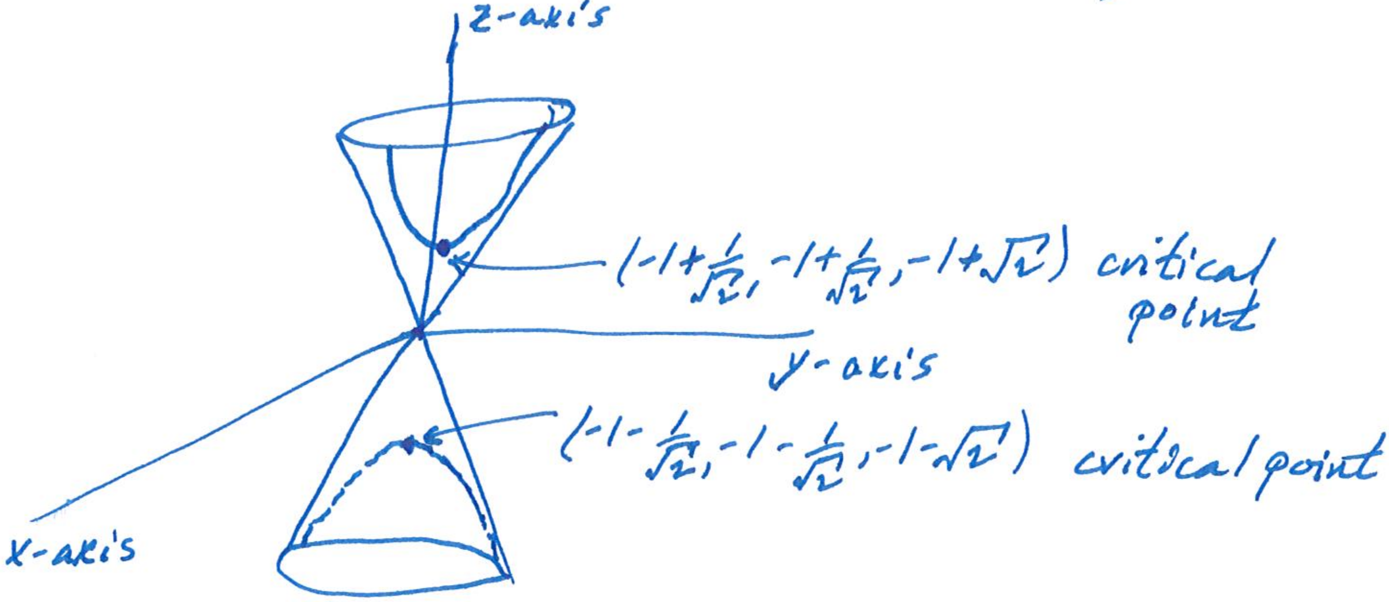
The picture on the following page illustrates why $(-1 - \frac{1}{\sqrt{2}}, -1 - \frac{1}{\sqrt{2}}, -1 - \sqrt{2})$ gives a local minimum, $(-1 + \frac{1}{\sqrt{2}}, -1 + \frac{1}{\sqrt{2}}, -1 + \sqrt{2})$ gives a global minimum, and

Ass 2 Q1 2.5 3

$f(x,y,z)$ has no maximum

The constraint $z^2=x^2+y^2$ is a cone and the constraint $z=1+x+y$ is a plane.

The intersection of these two is a hyperbola



On this hyperbola, the square of the distance to the origin,

$$f(x,y,z)=x^2+y^2+z^2=z^2+z^2=2z^2$$

gets arbitrarily large. So $f(x,y,z)$ has no maximum on the hyperbola.

(1) Constraints $z^2 = x^2+y^2$ and $z = 1+x+y$.

Minimse/Maximise $f(x,y,z) = x^2+y^2+z^2$.

Solution by reducing to two variables and one constraint:

Substitute $z = 1+x+y$ to $z^2 = x^2+y^2$ to get

$$(1+x+y)^2 = x^2+y^2.$$

So $1+x^2+y^2+2x+2y+2xy = x^2+y^2$

So $1+2x+2y+2xy = 0$. Let $g(x,y) = 1+2x+2y+2xy$.

Maximise/Minimise:

$$f(x,y) = x^2+y^2+z^2 = x^2+y^2+(1+x+y)^2$$
$$= 1+2x^2+2y^2+2x+2y+2xy$$
$$= 2x^2+2y^2+0 = 2x^2+2y^2.$$

$\nabla f = (4x, 4y)$ and $\nabla g = (2+2y, 2+2x)$

and $\nabla f = \lambda \nabla g$ gives
$$4x = \lambda(2+2y)$$
$$4y = \lambda(2+2x)$$

So $4(x-y) = \lambda(2+2y) - \lambda(2+2x) = 2\lambda y - 2\lambda x = 2\lambda(y-x)$.

So $2(x-y) = \lambda(y-x)$. So $2(x-y) = -\lambda(x-y)$.

So $(2+\lambda)(x-y) = 0$.

So $\lambda = -2$ or $x = y$.

If $x=y$ then the constraint gives

$$0 = 1+2x+2y+2xy = 1+2x+2x+2x^2 = 2x^2+4x+1$$

and so $0 = x^2+2x+\frac{1}{2} = (x^2+2x+1)-\frac{1}{2} = (x+1)^2 - \frac{1}{2}$.

So $x+1 = \pm\frac{1}{\sqrt{2}}$ and $x = -1 \pm \frac{1}{\sqrt{2}}$.

So critical points are at $x = -1\pm\frac{1}{\sqrt{2}}$, $x=y$, $z = 1+x+y$,

$$(x,y,z) = \left(-1+\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}, -1+\sqrt{2}\right)$$

$$(x,y,z) = \left(-1-\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, -1-\sqrt{2}\right).$$

For the first point $f(x,y,z) = 2(-1+\sqrt{2})^2 = 6-4\sqrt{2}$

For the second point $f(x,y,z) = 2(-1-\sqrt{2})^2 = 6+4\sqrt{2}$

So $(x,y,z) = \left(-1-\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, -1-\sqrt{2}\right)$ is a point on C local minimum from the origin (distance $\sqrt{6+4\sqrt{2}}$).

and $(x,y,z) = \left(-1+\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}, -1+\sqrt{2}\right)$ is the point on C closest to the origin (distance $\sqrt{6-4\sqrt{2}}$).

(Note: If $\lambda = -2$ then $4x = (-2)(2+2y) = -4-4y$.

So $x = -1-y$ and $x+y+1 = 0$.

Since $z = x+y+1$ then $z = 0$.

Since $z^2 = x^2+y^2$ then $x=0$ and $y=0$.

But this is a contradiction to $x+y+1=z$.

So this case cannot occur.)

Ass2Q1 5 1

(1) Constraints $z^2 = x^2+y^2$ and $z = 1+x+y$.
Minimize/Maximize $f(x,y,z) = x^2+y^2+z^2$.

Solution by reducing to a single variable:

Rewrite $z^2 = x^2+y^2$ as $x = z\cos\theta$
$y = z\sin\theta$

Then $z = 1+x+y = 1+z\cos\theta + z\sin\theta$

gives $z(1-\cos\theta-\sin\theta) = 1$.

So $z = \dfrac{1}{1-\cos\theta-\sin\theta}$

Minimize/Maximize

$f(x,y,z) = x^2+y^2+z^2 = z^2+z^2 = 2z^2$

$= 2\left(\dfrac{1}{1-\cos\theta-\sin\theta}\right)^2$

Critical points will be at $\dfrac{df}{d\theta} = 0$.

$\dfrac{df}{d\theta} = -4\left(\dfrac{1}{1-\cos\theta-\sin\theta}\right)^3(\sin\theta-\cos\theta)$

So $\dfrac{df}{d\theta}$ is zero when $\sin\theta - \cos\theta = 0$.

i.e. when $\sin\theta = \cos\theta$ and $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$.

If $\theta = \frac{\pi}{4}$ then $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$,

$z = \dfrac{1}{1-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}} = \dfrac{1}{1-\sqrt{2}}\dfrac{(1+\sqrt{2})}{(1+\sqrt{2})} = \dfrac{1+\sqrt{2}}{1-2} = -1-\sqrt{2}$

and $x = z\cos\theta = (-1-\sqrt{2})\frac{1}{\sqrt{2}} = -1 - \frac{1}{\sqrt{2}}$

and $y = z\sin\theta = (-1-\sqrt{2})\frac{1}{\sqrt{2}} = -1 - \frac{1}{\sqrt{2}}$.

So $(x,y,z) = (-1-\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, -1-\sqrt{2})$ and

$$f(x,y,z) = x^2+y^2+z^2 = 2z^2 = 2(1+2\sqrt{2}+2)$$
$$= 6+4\sqrt{2}$$

If $\theta = \frac{3\pi}{4}$ then $\sin\theta = \cos\theta = \frac{-1}{\sqrt{2}}$,

$$z = \frac{1}{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}} = \frac{1}{1+\sqrt{2}}\frac{(1-\sqrt{2})}{(1-\sqrt{2})} = \frac{1-\sqrt{2}}{1-2} = -1+\sqrt{2}$$

and $x = z\cos\theta = (-1+\sqrt{2})\left(\frac{-1}{\sqrt{2}}\right) = -1+\frac{1}{\sqrt{2}}$

and $y = z\sin\theta = (-1+\sqrt{2})\left(\frac{-1}{\sqrt{2}}\right) = -1+\frac{1}{\sqrt{2}}$.

So $(x,y,z) = (-1+\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}, -1+\sqrt{2})$ and

$$f(x,y,z) = x^2+y^2+z^2 = 2z^2 = 2(-1+\sqrt{2})^2$$
$$= 2(1-2\sqrt{2}+2) = 6-4\sqrt{2}.$$

So $(x,y,z) = (-1-\frac{1}{\sqrt{2}}, -1-\frac{1}{\sqrt{2}}, -1-\sqrt{2})$ is a point on C local minimum from the origin (distance $\sqrt{6+4\sqrt{2}}$) and $(x,y,z) = (-1+\frac{1}{\sqrt{2}}, -1+\frac{1}{\sqrt{2}}, -1+\sqrt{2})$ is the point on C closest to the origin (distance $\sqrt{6-4\sqrt{2}}$).

(1)

(2) Since $c(t) = (2(-t+\sin t), \sqrt{3}(1-\cos t), 1-\cos t)$

then

$$\frac{dc}{dt} = (2(-1+\cos t), \sqrt{3}\sin t, \sin t)$$

Since $\sin t = \sin(2\cdot\frac{t}{2}) = 2\sin\frac{t}{2}\cos\frac{t}{2}$

$\cos t = \cos(2\cdot\frac{t}{2}) = \cos^2\frac{t}{2} - \sin^2\frac{t}{2} = 1 - 2\sin^2\frac{t}{2}$

then $-1+\cos t = -2\sin^2\frac{t}{2}$ and

$$\frac{dc}{dt} = (-2\cdot 2\sin^2\frac{t}{2}, \sqrt{3}\cdot 2\sin\frac{t}{2}\cos\frac{t}{2}, 2\sin\frac{t}{2}\cos\frac{t}{2})$$

$$= 2\sin\frac{t}{2}(-2\sin\frac{t}{2}, \sqrt{3}\cos\frac{t}{2}, \cos\frac{t}{2})$$

Thus

$$\frac{ds}{dt} = \left|\frac{dc}{dt}\right| = \left(4\sin^2\frac{t}{2}\left(4\sin^2\frac{t}{2} + 3\cos^2\frac{t}{2} + \cos^2\frac{t}{2}\right)\right)^{\frac{1}{2}}$$

$$= 2\sin\frac{t}{2}\left(4(\sin^2\frac{t}{2} + \cos^2\frac{t}{2})\right)^{\frac{1}{2}} = 2\sin\frac{t}{2}\cdot 4^{\frac{1}{2}}$$

$= 4\sin\frac{t}{2}$ (which is positive for $0<t<2\pi$).

(a) The arc length along c for $0<t<2\pi$ is

$$s = \int_0^{2\pi}\left(\frac{ds}{dt}\right)dt = \int_0^{2\pi} 4\sin\left(\frac{t}{2}\right)dt = -4\cos\left(\frac{t}{2}\right)\cdot 2\Big|_0^{2\pi}$$

$$= -8(\cos\pi - \cos 0) = -8(-1-1) = 16$$

Ass 2 Q2 (8) (2)

(b) $T(t) = \dfrac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \dfrac{1}{4\sin\frac{t}{2}}\left(2\sin\frac{t}{2}\left(-2\sin\frac{t}{2}, \sqrt{3}\cos\frac{t}{2}, \cos\frac{t}{2}\right)\right)$

$= \frac{1}{2}\left(-2\sin\frac{t}{2}, \sqrt{3}\cos\frac{t}{2}, \cos\frac{t}{2}\right)$

$= \left(-\sin\frac{t}{2}, \frac{\sqrt{3}}{2}\cos\frac{t}{2}, \frac{1}{2}\cos\frac{t}{2}\right)$

So

$\dfrac{dT}{dt} = \left(-\frac{1}{2}\cos\frac{t}{2}, -\frac{\sqrt{3}}{4}\sin\frac{t}{2}, -\frac{1}{4}\sin\frac{t}{2}\right)$ and

$\left|\dfrac{dT}{dt}\right| = \left(\frac{1}{4}\cos^2\frac{t}{2} + \frac{3}{16}\sin^2\frac{t}{2} + \frac{1}{16}\sin^2\frac{t}{2}\right)^{1/2}$

$= \left(\frac{1}{4}\cos^2\frac{t}{2} + \frac{1}{4}\sin^2\frac{t}{2}\right)^{\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{2}.$

So $N(t) = \dfrac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = 2\left(-\frac{1}{2}\cos\frac{t}{2}, -\frac{\sqrt{3}}{4}\sin\frac{t}{2}, -\frac{1}{4}\sin\frac{t}{2}\right)$

$= \left(-\cos\frac{t}{2}, -\frac{\sqrt{3}}{2}\sin\frac{t}{2}, -\frac{1}{2}\sin\frac{t}{2}\right)$

Then

$B(t) = T(t) \times N(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\frac{t}{2} & \frac{\sqrt{3}}{2}\cos\frac{t}{2} & \frac{1}{2}\cos\frac{t}{2} \\ -\cos\frac{t}{2} & -\frac{\sqrt{3}}{2}\sin\frac{t}{2} & -\frac{1}{2}\sin\frac{t}{2} \end{vmatrix}$

Ass 2 Q 2 (9) (3)

$$= \hat{\imath}\left(-\frac{\sqrt{3}}{4}\sin\frac{t}{2}\cos\frac{t}{2} + \frac{\sqrt{3}}{4}\sin\frac{t}{2}\cos\frac{t}{2}\right)$$

$$+ \hat{\jmath}\left(\frac{1}{2}\sin^2\frac{t}{2} + \frac{1}{2}\cos^2\frac{t}{2}\right) + \hat{k}\left(\frac{\sqrt{3}}{2}\sin^2\frac{t}{2} + \frac{\sqrt{3}}{2}\cos^2\frac{t}{2}\right)$$

$$= 0\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{\sqrt{3}}{2}\hat{k} = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

(c) $$K(t) = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{\frac{1}{2}}{4\sin\frac{t}{2}} = \frac{1}{8\sin(\frac{t}{2})}$$

Since $B(t) = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ then

$$\frac{dB}{ds} = \frac{\frac{dB}{dt}}{\frac{ds}{dt}} = \frac{0}{4\sin\frac{t}{2}} = 0. \quad \left(\text{since } B \text{ is a constant}\right)$$

Since $\tau(t)$ is such that $\frac{dB}{ds} = -\tau(t)N(t)$

then $\tau(t) = 0$.

This indicates that the curve lies in a plane (the plane $y = \sqrt{3}z$).

This is the plane

$0x + y - \sqrt{3}z = 0$

Ass 2 Q3 (10) (1)

(3) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

Since $\vec{F} = (e^x\cos y + yz)\,\hat{i} + (xz - e^x\sin y)\,\hat{j} + (xy+z)\,\hat{k}$

then

$\vec{\nabla}\times\vec{F} = \hat{i}(x-x) + \hat{j}(y-y) + \hat{k}(z-z) = 0.$

So $\vec{F}$ is irrotational.

If

$f(x,y,z) = e^x\cos y + xyz + \frac{1}{2}z^2$

then

$\frac{\partial f}{\partial x} = e^x\cos y + yz$

$\frac{\partial f}{\partial y} = -e^x\sin y + xz$ and so $\vec{F} = \vec{\nabla} f$

$\frac{\partial f}{\partial z} = xy + z$