Vector Calculus MAST20009 Solutions Ass1Q1

Assignment 1 Question 1

$$f(x,y) = \begin{cases} \dfrac{3x^2-y^2-2}{x^2+y^2-2}, & \text{for } x^2+y^2 \neq 2 \\ -1, & \text{for } x^2+y^2 = 2 \end{cases}$$

(a) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2-2}{x^2+y^2-2}$.

Using the limit theorems,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2-2}{x^2+y^2-2} = \frac{\displaystyle\lim_{(x,y)\to(0,0)} (3x^2-y^2-2)}{\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2-2)}$$

$$= \frac{3\left(\displaystyle\lim_{(x,y)\to(0,0)} x\right)^2 - \left(\displaystyle\lim_{(x,y)\to(0,0)} y\right)^2 - \displaystyle\lim_{(x,y)\to(0,0)} 2}{\left(\displaystyle\lim_{(x,y)\to(0,0)} x\right)^2 + \left(\displaystyle\lim_{(x,y)\to(0,0)} y\right)^2 - \displaystyle\lim_{(x,y)\to(0,0)} 2}$$

$$= \frac{3\cdot 0^2 - 0^2 - 2}{0^2+0^2-2} = \frac{-2}{-2} = 1.$$

(b) Evaluate $\displaystyle \lim_{(x,y)\to(1,1)} \frac{3x^2-y^2-2}{x^2+y^2-2}$

$$\lim_{\substack{(x,y)\to(1,1)\\ x=1}} \frac{3x^2-y^2-2}{x^2+y^2-2} = \lim_{y\to 1} \frac{3-y^2-2}{1+y^2-2}$$

$$= \lim_{y\to 1} \frac{1-y^2}{y^2-1} = \lim_{y\to 1} -1 = -1$$

and

$$\lim_{\substack{(x,y)\to(1,1)\\ y=1}} \frac{3x^2-y^2-2}{x^2+y^2-2} = \lim_{x\to 1} \frac{3x^2-1-2}{x^2+1-2}$$

$$= \lim_{x\to 1} \frac{3x^2-3}{x^2-1} = \lim_{x\to 1} 3 = 3$$

Since the limit approaches different values from different directions

$$\lim_{(x,y)\to(1,1)} \frac{3x^2-y^2-2}{x^2+y^2-2} \text{ does not exist.}$$

(c) Is f continuous at $(1,1)$?

The question asks: Is $\displaystyle \lim_{(x,y)\to(1,1)} f(x,y) = f(1,1)$?

Since $1^2+1^2=2$ then $f(1,1) = -1$.

By part (b), $\displaystyle \lim_{(x,y)\to(1,1)} f(x,y)$ does not exist

so f is <u>not</u> continuous at $(1,1)$.

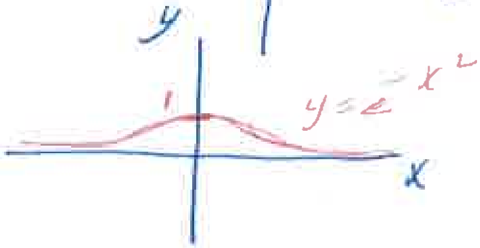
Assignment 1 Question 2

$$g(x,y) = \begin{cases} e^{-(x^2+y^2)} - 5, & \text{for } (x,y) \neq (0,0) \\ 1, & \text{for } (x,y) = (0,0) \end{cases}$$

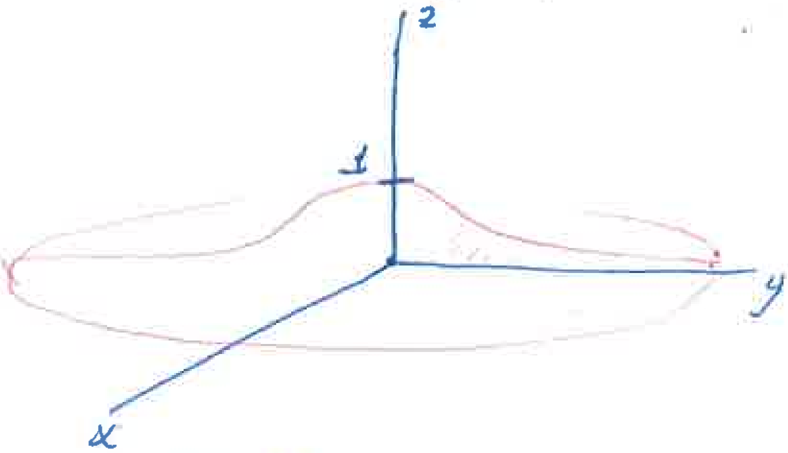
(a) Graph $g(x,y)$.

Since $y = e^x$ has graph

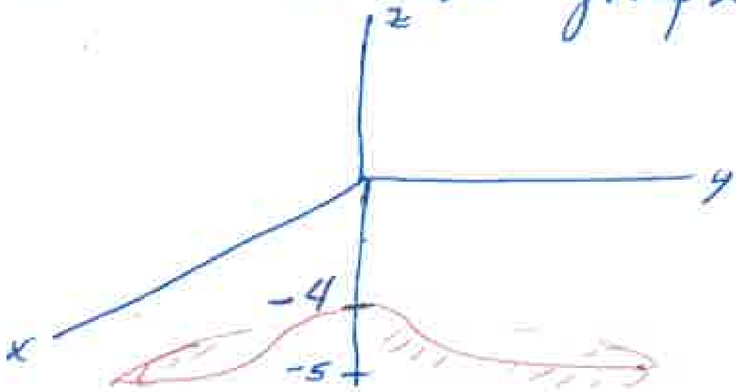
then $y = e^{-x^2}$ has graph



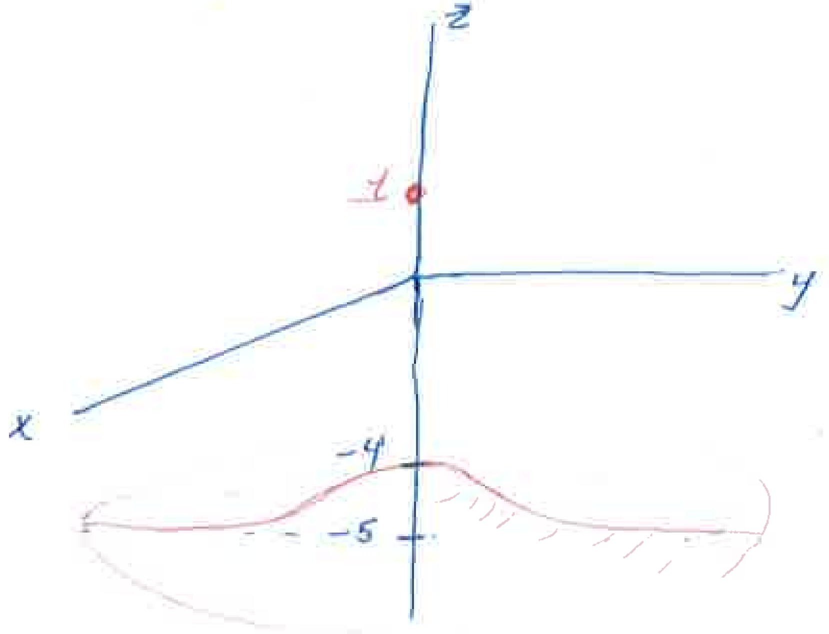
and $z = e^{-(x^2+y^2)}$ has graph



and $z = e^{-(x^2+y^2)} - 5$ has graph



Thus $z = g(x,y)$ has graph



(d) Calculate $\lim_{(x,y)\to(0,0)} g(x,y)$.

Since e^z is continuous

$$\lim_{(x,y)\to(0,0)} e^{-(x^2+y^2)} = e^{-\lim_{(x,y)\to(0,0)}(x^2+y^2)} = e^{-0} = 1.$$

By the limit theorems

$$\lim_{(x,y)\to(0,0)} e^{-(x^2+y^2)} - 5 = \lim_{(x,y)\to(0,0)} e^{-(x^2+y^2)} - \lim_{(x,y)\to(0,0)} 5$$

$$= 1 - 5 = -4.$$

So

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\neq(0,0)}} g(x,y) = -4.$$

(c) Determine where g is continuous.

Since
$-(x^2+y^2)$ is a polynomial it is continuous for $(x,y) \in \mathbb{R}^2$

Since
e^z is continuous for $z \in \mathbb{R}$ then
$e^{-(x^2+y^2)}$ is continuous for $(x,y) \in \mathbb{R}^2$

Since $z-5$ is a polynomial, it is continuous for $z \in \mathbb{R}$.

So $e^{-(x^2+y^2)} - 5$ is continuous for $(x,y) \in \mathbb{R}^2$

So $\boxed{g(x,y) \text{ is continuous for } (x,y) \neq (0,0).}$

Is $g(x,y)$ continuous at $(0,0)$?

$$\lim_{(x,y)\to(0,0)} g(x,y) \stackrel{?}{=} g(0,0).$$

We know $g(0,0) = 1$ and

$$\lim_{(x,y)\to(0,0)} g(x,y) = -4 \quad \text{(from part (b))}.$$

So $\boxed{g(x,y) \text{ is } \underline{\text{not}} \text{ continuous at } (0,0).}$

(d) Find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ for $(x,y) \neq (0,0)$.

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(e^{-(x^2+y^2)} - 5\right) = -2x e^{-(x^2+y^2)},$$

for $(x,y) \neq (0,0)$.

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(e^{-(x^2+y^2)} - 5\right) = -2y e^{-(x^2+y^2)}$$

for $(x,y) \neq (0,0)$

(e) No matter how many times we apply $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ (in any combination) the result will always be (for $(x,y) \neq (0,0)$)

$$q_r(x,y) e^{-(x^2+y^2)}$$

where $q_r(x,y)$ is a polynomial.

Since polynomials are continuous for $(x,y) \in \mathbb{R}^2$ and $e^{-(x^2+y^2)}$ (an exponential of a polynomial) is continuous for $(x,y) \in \mathbb{R}^2$ then

all partial derivatives of $g(x,y)$ are continuous at $(0,1)$.

So f is C^r at $(0,1)$ for all $r \in \mathbb{Z}_{\geq 0}$.

Assignment 1 Question 3

$f(u,v) = (2u^2, 3v, v^2 - u)$, $g(u,v,w) = (v + w^2, u^2 + w)$

$h(u,v) = (v^2 - u, 2u + v)$

Evaluate $\underline{\underline{D}}(h(g(f(u,v))))$ at $(u,v) = (0,1)$.

$$\left.\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix}\right|_{(u,v)=(0,1)} = \left.\begin{pmatrix} 4u & 0 \\ 0 & 3 \\ -1 & 2v \end{pmatrix}\right|_{(u,v)=(0,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \\ -1 & 2 \end{pmatrix}$$

$f(0,1) = (2\cdot 0^2, 3\cdot 1, 1^2 - 0) = (0, 3, 1)$.

$$\left.\begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} & \frac{\partial g_1}{\partial w} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} & \frac{\partial g_2}{\partial w} \end{pmatrix}\right|_{(u,v,w)=(0,3,1)} = \left.\begin{pmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \end{pmatrix}\right|_{(u,v,w)=(0,3,1)} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$g(0,3,1) = (3 + 1^2, 0^2 + 1) = (4, 1)$

$$\left.\begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}\right|_{(u,v)=(4,1)} = \left.\begin{pmatrix} -1 & 2v \\ 2 & 1 \end{pmatrix}\right|_{(u,v)=(4,1)} = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

So

$D(h(g(f(u,v))))$ at $(u,v)=(0,1)$ is

$$\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 6 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 12 & -1 & 0 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 \\ -5 & 16 \end{pmatrix}$$