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Vector Calculus Lecture 11

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Uni Melb
A. Ram

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§2.2 Example 7 Show that

$$c(t) = (\sin t, \cos t, 2t)$$

is a flow line of $\vec{F}(x,y,z) = (y, -x, 2)$

Solution:

$$c'(t) = \frac{d\vec{c}}{dt} = (\cos t, -\sin t, 2)$$

$$\vec{F}(\vec{c}(t)) = \vec{F}(\sin t, \cos t, 2t)$$

$$= (\cos t, -\sin t, 2).$$

Since $c'(t) = \vec{F}(\vec{c}(t))$ then $\vec{c}(t)$ is a flow line of $\vec{F}$.

§2.2 Example 8 Let $\vec{F}(x,y) = (-y, x)$

Sketch the flow lines and determine their equation.

Solution If $\vec{c}^{*}(t) = (x(t), y(t))$ is a flow line then

$$\frac{d\vec{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-y, x) = \vec{F}.$$

So

$$\frac{dx}{dt} = -y \qquad \text{So} \quad \frac{dy}{dx} = -\frac{x}{y}. \quad \text{So} \quad \int y\,dy = \int -x\,dx$$

$$\frac{dy}{dt} = x$$

So $\frac{y^2}{2} = -\frac{x^2}{2} + c$. So $\frac{1}{2}(x^2+y^2) = c$

So the flow lines are circles. See p. 94.

§2.3 Example 1 Compute $\vec{\nabla}\cdot\vec{F}$ when

$$\vec{F} = x^2y\,\hat{i} + z\,\hat{j} + xyz\,\hat{k}.$$

Solution:

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial x^2y}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial xyz}{\partial z}$$

$$= 2xy + 0 + xy = 3xy.$$

§2.3 Example 2 Compute $\vec{\nabla}\cdot\vec{F}$ when

$$\vec{F} = x\,\hat{i}.$$

Solution: $\vec{\nabla}\cdot\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z} = 1 + 0 + 0 = 1$

since $\vec{F} = x\hat{i} + 0\hat{j} + 0\hat{k}$.

§2.3 Example 3 Compute $\vec{\nabla}\cdot\vec{F}$ when

$$\vec{F} = -y\,\hat{j}$$

Solution: Since $\vec{F} = 0\hat{i} - y\hat{j} + 0\hat{k}$ then

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial 0}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial 0}{\partial z} = 0 + (-1) + 0 = -1.$$

§2.3 Example 4 Compute $\vec{\nabla}\cdot\vec{F}$ when

$$\vec{F} = x\hat{i} - y\hat{j} + 0\hat{k}.$$

Solution: $\vec{\nabla}\cdot\vec{F} = \frac{\partial x}{\partial x} + \frac{\partial(-y)}{\partial y} + \frac{\partial 0}{\partial z} = 1 + (-1) + 0 = 0$

§2.3 Example 5 Find $\vec{\nabla}\times\vec{F}$ when

$$\vec{F} = x^2y\hat{i} - 2xz\hat{j} + (x+y-z)\hat{k}.$$

Solution

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & x+y-z \end{vmatrix}$$

$$= \hat{i}(1-(-2x)) - \hat{j}(1-0) + \hat{k}(-2z-x^2)$$

$$= (1+2x)\hat{i} - \hat{j} - (x^2+2z)\hat{k}$$

§2.3 Example 6 Is $\vec{F}$ irrotational when

$$\vec{F} = x\hat{i} - y\hat{j}\,?$$

Solution:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = \begin{matrix} \hat{i}(0-0) \\ -\hat{j}(0-0) \\ +\hat{k}(0-0) \end{matrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

So $\vec{F}$ is irrotational.

§2.3 Example 7 Is $\vec{F}$ irrotational when

$$\vec{F} = -y\hat{i} + x\hat{j}$$

Solution:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{matrix} \hat{i}(0-0) \\ -\hat{j}(0-0) \\ +\hat{k}(1-(-1)) \end{matrix} = 2\hat{k} \neq 0.$$

So $\vec{F}$ is *not* irrotational.

§2.3 Example 8 Find $\nabla^2 f$ when

$$f = x^2y + xy^2 + yz^2.$$

Solution:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2xy + y^2) + \frac{\partial}{\partial y}(x^2 + 2xy + z^2) + \frac{\partial}{\partial z}(0 + 0 + 2yz)$$

$$= (2y + 0) + (0 + 2x + 0) + 2y$$

$$= 2x + 4y.$$

§2.3 Example 9 Find $\nabla^2 \vec{F}$ when

$$\vec{F} = x^2y\,\hat{i} + xy^2z\,\hat{j} + y^3z^2\,\hat{k}$$

Solution:

$$\nabla^2 \vec{F} = \nabla^2(x^2y)\,\hat{i} + \nabla^2(xy^2z)\,\hat{j} + \nabla^2(y^3z^2)\,\hat{k}$$

$$= \left(\frac{\partial^2(x^2y)}{\partial x^2} + \frac{\partial^2(x^2y)}{\partial y^2} + \frac{\partial^2(x^2y)}{\partial z^2}\right)\hat{i}$$

$$+ \left(\frac{\partial^2(xy^2z)}{\partial x^2} + \frac{\partial^2(xy^2z)}{\partial y^2} + \frac{\partial^2(xy^2z)}{\partial z^2}\right)\hat{j}$$

$$+ \left(\frac{\partial^2(y^3z^2)}{\partial x^2} + \frac{\partial^2(y^3z^2)}{\partial y^2} + \frac{\partial^2(y^3z^2)}{\partial z^2}\right)\hat{k}$$

$$= \left(\frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}x^2 + 0\right)\hat{i} + \left(\frac{\partial}{\partial x}(y^2z) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(xy^2)\right)\hat{j}$$

$$+ \left(0 + \frac{\partial}{\partial y}(3y^2z^2) + \frac{\partial}{\partial z}(2y^3z)\right)\hat{k}$$

$$= (2y + 0 + 0)\hat{i} + (0 + 2xz + 0)\hat{j}$$
$$+ (0 + 6yz^2 + 2y^3)\hat{k}$$
$$= 2y\hat{i} + 2xz\hat{j} + (6yz^2 + 2y^3)\hat{k}.$$