Vector calculus Lecture 12

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grad, curl and div: The Koszul complex for $\mathbb{R}^3$.

The space of functions on $\mathbb{R}^3$ is

$$\mathcal{O}_{\mathbb{R}^3} = \{ f: \mathbb{R}^3 \to \mathbb{R} \}$$

The space of 1-forms on $\mathbb{R}^3$ is

$$\Omega^1_{\mathbb{R}^3} = \left\{ \vec{F} = F_1 dx + F_2 dy + F_3 dz \;\middle|\; \begin{matrix} F_1, F_2, F_3 \\ \text{are functions} \end{matrix} \right\}$$

The space of 2-forms on $\mathbb{R}^3$ is

$$\Omega^2_{\mathbb{R}^3} = \left\{ \vec{G} = G_1 dy\wedge dz + G_2 dx\wedge dz + G_3 dx \wedge dy \;\middle|\; \begin{matrix} G_1, G_2, G_3 \\ \text{are functions} \end{matrix} \right\}$$

The space of 3-forms on $\mathbb{R}^3$ is

$$\Omega^3_{\mathbb{R}^3} = \{ H\, dx\wedge dy \wedge dz \mid H \text{ is a function} \}.$$

The Koszul complex for $\mathbb{R}^3$ is the sequence of maps

$$\mathcal{O}_{\mathbb{R}^3} \xrightarrow{\text{grad}} \Omega^1_{\mathbb{R}^3} \xrightarrow{\text{curl}} \Omega^2_{\mathbb{R}^3} \xrightarrow{\text{div}} \Omega^3_{\mathbb{R}^3}$$

where

$$\mathrm{grad}: \mathcal{O}_{\mathbb{R}^3} \longrightarrow \Omega^1_{\mathbb{R}^3}$$

$$\mathrm{curl}: \Omega^1_{\mathbb{R}^3} \longrightarrow \Omega^2_{\mathbb{R}^3} \quad \text{and}$$

$$\mathrm{div}: \Omega^2_{\mathbb{R}^3} \longrightarrow \Omega^3_{\mathbb{R}^3}$$

are given by

$$\mathrm{grad}(f) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

$$\begin{aligned}\mathrm{curl}(F_1dx + F_2dy + F_3dz) &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy\wedge dz \\ &\quad - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)dx\wedge dz \\ &\quad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\wedge dy,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{div}(G_1\,dy\wedge dz + G_2\,dx\wedge dz + G_3\,dx\wedge dy) \\ &\quad = \left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z}\right)dx\wedge dy\wedge dz.\end{aligned}$$

<u>Theorem</u> (grad, curl and div define a complex)

(a) $\mathrm{curl}(\mathrm{grad}(f)) = 0$

(b) $\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$

<u>Theorem</u> (The Koszul complex is exact)

(a) If $\mathrm{curl}(\vec{F}) = 0$ then there exists f such that $\vec{F} = \mathrm{grad}(f)$.

(b) If $\mathrm{div}(\vec{G}) = 0$ then there exists $\vec{F}$ such that $\vec{G} = \mathrm{curl}(\vec{F})$.

<u>Definitions</u>

- $\vec{G}$ is <u>incompressible</u>, or <u>a closed 2-form</u>, if $\mathrm{div}(\vec{G}) = 0$
- $\vec{G}$ is an <u>exact 2-form</u> or <u>vector potential</u> if there exists $\vec{F}$ such that $\vec{G} = \mathrm{curl}(\vec{F})$
- $\vec{F}$ is <u>irrotational</u>, or <u>a closed 1-form</u>, if $\mathrm{curl}(\vec{F}) = 0$.
- $\vec{F}$ is an <u>exact 1-form</u> or <u>gradient field</u> if there exists f such that $\vec{F} = \mathrm{grad}(f)$

§2.4 Example 1 Show that curl(grad(f)) = 0.

Solution: The problem asks to show that

$$\vec{\nabla} \times (\vec{\nabla} f) = 0.$$

$$\vec{\nabla} \times (\vec{\nabla} f) = \vec{\nabla} \times \left(\frac{\partial f}{\partial x} \hat{\imath} + \frac{\partial f}{\partial y} \hat{\jmath} + \frac{\partial f}{\partial z} \hat{k} \right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{\imath} - \hat{\jmath} \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) + \hat{k} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= 0 \hat{\imath} + 0 \hat{\jmath} + 0 \hat{k}.$$

§2.4 Example 3 Show that div(curl(F)) = 0.

Solution: The problem asks to show that

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0.$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \vec{\nabla} \cdot \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\imath} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \hat{\jmath} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0.$$