Vector Calculus 2018 Lecture 13

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§2.4 Example 2 Let

$$\vec{V} = (2xy^2, 2x^2y + 2z^2y, 2y^2z).$$

Is $\vec{V}$ a gradient field?

Solution: The question is

Is there $f$ such that $\vec{\nabla} f = \vec{V}$?

Since

$$\mathrm{curl}(\vec{V}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{\imath}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \hat{\imath}(4yz - (0 + 4yz)) - \hat{\jmath}(0 - 0) + \hat{k}(4xy + 0 - (4xy))$$

$$= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 0,$$

then $f$ should exist. If $\vec{\nabla} f = \vec{V}$ then

$$\frac{\partial f}{\partial x} = 2xy^2, \quad \frac{\partial f}{\partial y} = 2x^2y + 2z^2y, \quad \frac{\partial f}{\partial z} = 2y^2z$$

Guess: $f = x^2y^2 + y^2z^2$

Check: $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(x^2y^2 + y^2z^2) = 2xy^2 + 0 = 2xy^2$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y^2+y^2z^2) = 2x^2y + 2yz^2$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2y^2+y^2z^2) = 0 + 2y^2z = 2y^2z$$

So $\vec{\nabla} f = \vec{V}$.

So $\vec{V}$ is a gradient field.

__§2.4 Example 4__ Let

$$\vec{V} = (x^2+1, z-2xy, y).$$

Find $\vec{F}$ so that $\text{curl}(\vec{F}) = \vec{V}$.

__Solution__: Since

$$\text{div}(\vec{V}) = \vec{\nabla}\cdot\vec{V} = \frac{\partial(x^2+1)}{\partial x} + \frac{\partial(z-2xy)}{\partial y} + \frac{\partial y}{\partial z}$$

$$= 2x + (0-2x) + 0 = 0$$

then $\vec{F}$ should exist.

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$.

Then

$$\operatorname{curl}(\vec{F}) = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

This equals $\vec{V}$ when

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = x^2+1,$$

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = z - 2xy,$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y,$$

If we are really lucky, maybe it will work with $F_3 = 0$.

If $F_3 = 0$ then

$$\frac{\partial F_2}{\partial z} = -x^2-1, \qquad F_2 = (-x^2-1)z + g_2(x,y)$$

$$\frac{\partial F_1}{\partial z} = z - 2xy, \quad \text{and} \quad F_1 = \tfrac{1}{2}z^2 - 2xyz + g_1(x,y).$$

Then

$$y = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \left(-2xz + \frac{\partial g_2}{\partial x}\right) - \left(-2xz + \frac{\partial g_1}{\partial y}\right)$$

$$= \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}.$$

This will work if $g_2 = xy$ and $g_1 = 0$.

So, we think $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$F_1 = \tfrac{1}{2}z^2 - 2yzx,$$
$$F_2 = -x^2 z - z + xy, \qquad \text{should work.}$$
$$F_3 = 0,$$

Check: $\dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} = 0 - (-x^2 - 1 + 0) = x^2 + 1$

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = (z - 2yx) - 0 = z - 2xy$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \cancel{(0 - z} (-2xz - 0 + y) - (0 - 2zx)$$
$$= y.$$

So $\mathrm{curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = V$ when

$$\vec{F} = (\tfrac{1}{2}z^2 - 2yzx)\hat{i} + (-x^2 z - z + xy)\hat{j} + 0\hat{k}.$$

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§2.4 Example 5 Let

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \quad \text{and} \quad \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

Find $\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^2}\right)$

Solution: Using the product rule for $\vec{\nabla}\cdot$

$$\vec{\nabla}\cdot\left(\frac{1}{r^2}\vec{r}\right) = \frac{1}{r^2}\vec{\nabla}\cdot\vec{r} + \vec{\nabla}\left(\frac{1}{r^2}\right)\cdot\vec{r}$$

$$= \frac{1}{r^2}\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$+ \vec{\nabla}\left((x^2+y^2+z^2)^{-1}\right)\vec{r}$$

$$= \frac{1}{r^2}\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\right)$$

$$+\Big((-1)(x^2+y^2+z^2)^{-2}2x\hat{i}+(-1)(x^2+y^2+z^2)^{-2}2y\hat{j} + (-1)(x^2+y^2+z^2)^{-2}2z\hat{k}\Big)\cdot\vec{r}$$

$$= \frac{1}{r^2}\cdot 3 + \frac{-2}{(x^2+y^2+z^2)^2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$= \frac{1}{r^2}\cdot 3 + \frac{-2}{(x^2+y^2+z^2)^2}(x^2+y^2+z^2)$$

$$= \frac{3}{r^2} - \frac{2}{x^2+y^2+z^2} = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

§2.4 Example 6 Let

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \quad \text{and} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

Find $\nabla^2 (r^2 \log r)$

Solution: Use the identity

$$\nabla^2 (fg) = \vec{\nabla}\cdot\vec{\nabla}(fg) = \vec{\nabla}\cdot(g\vec{\nabla}f + f\vec{\nabla}g)$$

$$= \vec{\nabla}\cdot(g\vec{\nabla}f) + \vec{\nabla}\cdot(f\vec{\nabla}g)$$

$$= \vec{\nabla}g\cdot\vec{\nabla}f + g(\vec{\nabla}\cdot\vec{\nabla}f) + \vec{\nabla}f\cdot\vec{\nabla}g + f(\vec{\nabla}\cdot\vec{\nabla}g)$$

$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\,\vec{\nabla}f\cdot\vec{\nabla}g.$$

So

$$\nabla^2 (r^2\log r) = r^2\,\nabla^2 \log r + \log r\,\nabla^2 r^2 + 2(\vec{\nabla}r^2)\cdot(\vec{\nabla}\log r)$$

Then

$$\vec{\nabla}r^2 = \vec{\nabla}(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 2\vec{r}.$$

$$\nabla^2 r^2 = \vec{\nabla}\cdot\vec{\nabla}r^2 = \vec{\nabla}\cdot 2\vec{r} = \frac{\partial(2x)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(2z)}{\partial z}$$

$$= 2+2+2 = 6.$$

$$\vec{\nabla}\log r = \frac{\partial \log r}{\partial x}\hat{i} + \frac{\partial \log r}{\partial y}\hat{j} + \frac{\partial \log r}{\partial z}\hat{k}$$

$$= \frac{1}{r}\frac{\partial r}{\partial x}\hat{i} + \frac{1}{r}\frac{\partial r}{\partial y}\hat{j} + \frac{1}{r}\frac{\partial r}{\partial z}\hat{k}$$

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$$= \frac{1}{r}\left( \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} 2x\,\hat{i} + \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} 2y\,\hat{j} + \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}} 2z\,\hat{k} \right)$$

$$= \frac{1}{r} \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} (x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{1}{r}\frac{1}{r}\vec{r} = \frac{1}{r^2}\vec{r}.$$

$$\nabla^2 \log r = \vec{\nabla}\cdot\vec{\nabla}\log r = \vec{\nabla}\cdot\left(\frac{1}{r^2}\vec{r}\right)$$

$$= \vec{\nabla}\left(\frac{1}{r^2}\right)\cdot\vec{r} + \frac{1}{r^2}\vec{\nabla}\cdot\vec{r}$$

$$= \left(\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1}\hat{i} + \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1}\hat{j} + \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1}\hat{k}\right) \cdot (x\hat{i}+y\hat{j}+z\hat{k})$$

$$+ \frac{1}{r^2}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)$$

$$= \left((-1)(x^2+y^2+z^2)^{-2} 2x\,\hat{i} + (-1)(x^2+y^2+z^2)^{-2} 2y\,\hat{j} + (-1)(x^2+y^2+z^2)^{-2} 2z\,\hat{k}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$+ \frac{1}{r^2} 3$$

$$= \frac{-2}{(x^2+y^2+z^2)^2}(x^2+y^2+z^2) + \frac{1}{r^2} 3$$

$$= \frac{-2}{x^2+y^2+z^2} + \frac{1}{r^2} 3 = \frac{-2}{r^2} + \frac{3}{r^2} = \frac{1}{r^2}$$

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So

$$\nabla^2(r^2\log r) = r^2\nabla^2\log r + \log r\,\nabla^2 r^2 + 2(\vec{\nabla} r^2)\cdot(\vec{\nabla}\log r)$$

$$= r^2\frac{1}{r^2} + \log r\cdot 6 + 2(2\vec{r})\cdot(\frac{1}{r^2}\vec{r})$$

$$= 1 + 6\log r + \frac{4}{r^2}\,\vec{r}\cdot\vec{r}$$

$$= 1 + 6\log r + \frac{4}{r^2}\,r^2$$

$$= 1 + 6\log r + 4 = 5 + 6\log r.$$

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§2.4 Example 7 Show that

$$\vec{\nabla}\cdot(f\vec{\nabla}g - g\vec{\nabla}f) = f\,\nabla^2 g - g\,\nabla^2 f$$

Solution:

$$\vec{\nabla}\cdot(f\vec{\nabla}g - g\vec{\nabla}f) = \vec{\nabla}\cdot(f\vec{\nabla}g) - \vec{\nabla}\cdot(g\vec{\nabla}f)$$

$$= (\vec{\nabla}f)\cdot(\vec{\nabla}g) + f(\vec{\nabla}\cdot\vec{\nabla}g)$$

$$- (\vec{\nabla}g)\cdot(\vec{\nabla}f) - g(\vec{\nabla}\cdot\vec{\nabla}f)$$

$$= f(\nabla^2 g) - g(\nabla^2 f)$$