Vector Calculus Lect. 16

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UniMelb.
A. Ram.

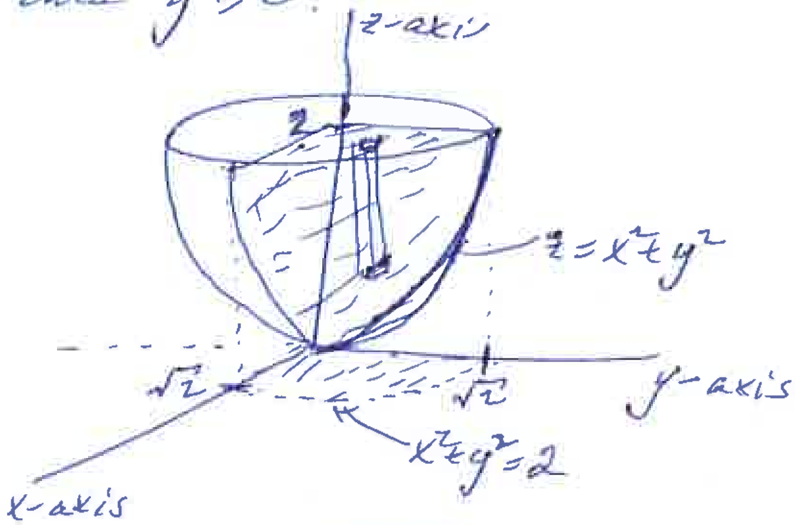
§3.2 Example 6 Evaluate $\iiint_D x\,dV$, where

D is the solid region bounded by

$x=0, \quad y=0, \quad z=2, \quad \text{and} \quad z=x^2+y^2$

and $x \geq 0$ and $y \geq 0$.

Solution:



$$\iiint_D x\,dV = \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} \int_{z=x^2+y^2}^{z=2} x \, dz\,dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} xz\Big]_{z=x^2+y^2}^{z=2} dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} \left(2x-(x^2+y^2)x\right) dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=\sqrt{2-x^2}} (2x-x^3-y^2x)\, dy\,dx$$

$$= \int_{x=0}^{x=\sqrt{2}} 2xy - x^3y - \frac{y^3}{3}x \Big]_{y=0}^{y=\sqrt{2-x^2}} dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \left(2x(2-x^2)^{\frac{1}{2}} - x^3(2-x^2)^{\frac{1}{2}} + \frac{1}{3}(2-x^2)^{3/2} x \right) dx$$

$$= \int_{x=0}^{x=\sqrt{2}} x(2-x^2)(2-x^2)^{\frac{1}{2}} + \frac{1}{3}(2-x^2)^{3/2} x \quad dx$$

$$= \int_{x=0}^{x=\sqrt{2}} \left(1 + \frac{1}{3}\right) x (2-x^2)^{3/2} dx = \int_{x=0}^{x=\sqrt{2}} \frac{4}{3} x (2-x^2)^{3/2} dx$$

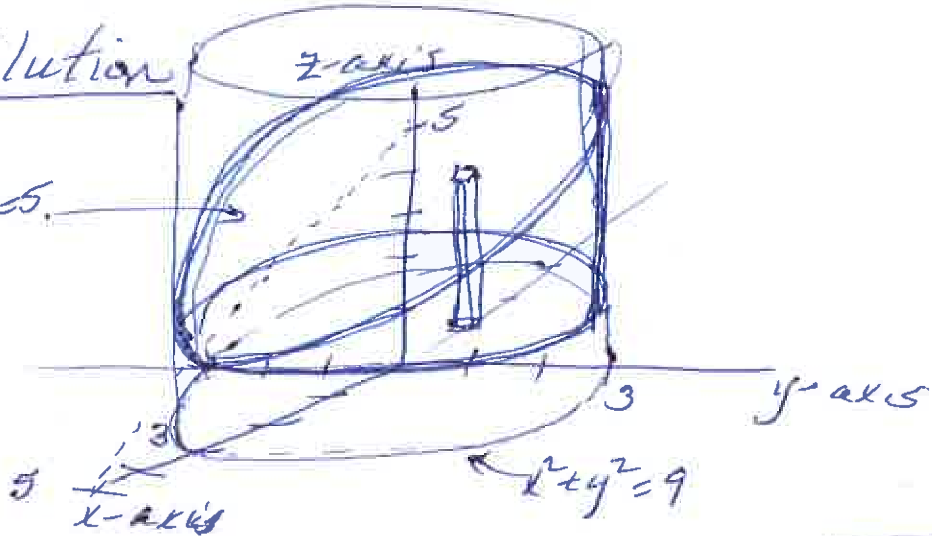
$$= (2-x^2)^{5/2} \cdot \frac{2}{5} \frac{1}{(-2)} \frac{4}{3} \Big]_{x=0}^{x=\sqrt{2}}$$

$$= \frac{-4}{15}(2-x^2)^{5/2} \Big]_{x=0}^{x=\sqrt{2}} = \frac{-4}{15} \cdot (2-2)^{5/2} - \left(\frac{-4}{15}(2-0)^{5/2} \right)$$

$$= 0 + \frac{4}{15} 2^{5/2} = \frac{4}{15} \cdot 2^{4/2} 2^{\frac{1}{2}} = \frac{16}{15}\sqrt{2} = \frac{8\sqrt{2}}{15}.$$

§3.2 Example 7 Find the volume of the solid region enclosed by the cylinder $x^2+y^2=9$ and the planes $z=1$, $x+z=5$.

Solution



$$\text{Volume} = \iiint_D dV = \int_{x=-3}^{x=3}\int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}}\int_{z=1}^{z=5-x} dz\,dy\,dx$$

$$= \int_{x=-3}^{x=3}\int_{y=-(9-x^2)^{\frac{1}{2}}}^{y=(9-x^2)^{\frac{1}{2}}} z\Big]_{z=1}^{z=5-x} dy\,dx$$

$$= \int_{x=-3}^{x=3}\int_{y=-(9-x^2)^{\frac{1}{2}}}^{y=(9-x^2)^{\frac{1}{2}}} ((5-x)-1)\,dy\,dx$$

$$= \int_{x=-3}^{x=3}\int_{y=-(9-x^2)^{\frac{1}{2}}}^{y=(9-x^2)^{\frac{1}{2}}} (4-x)\,dy\,dx$$

$$= \int_{x=-3}^{x=3} (4-x)y\Big]_{-(9-x^2)^{\frac{1}{2}}}^{(9-x^2)^{\frac{1}{2}}} dx$$

$$= \int_{x=-3}^{x=3} \left((4-x)(9-x^2)^{\frac{1}{2}} - (4-x)\left(-(9-x^2)^{\frac{1}{2}}\right)\right) dx$$

$$= \int_{x=-3}^{x=3} 2(4-x)(9-x^2)^{\frac{1}{2}} dx$$

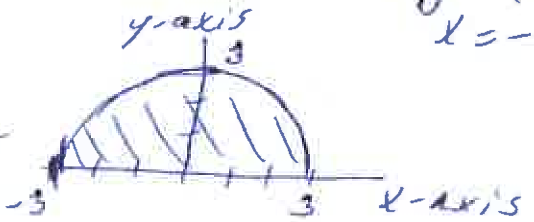
$$= \int_{x=-3}^{x=3} 8(9-x^2)^{\frac{1}{2}} - 2x(9-x^2)^{\frac{1}{2}} dx$$

$$= \int_{x=-3}^{x=3} 8(9-x^2)^{\frac{1}{2}} dx + \int_{x=-3}^{x=3} (-2x)(9-x^2)^{\frac{1}{2}} dx$$

$$= \int_{x=-3}^{x=3} 8(9-x^2)^{\frac{1}{2}} dx + \left(\left[\tfrac{2}{3}(9-x^2)^{3/2}\right]_{x=-3}^{x=3} \right)$$

$$= \int_{x=-3}^{x=3} 8(9-x^2)^{\frac{1}{2}} dx + \left(\tfrac{2}{3}(9-9)^{3/2} - \tfrac{2}{3}(9-9)^{3/2} \right)$$

$$= \int_{x=-3}^{x=3} 8(9-x^2)^{\frac{1}{2}} dx + 0 = 8 \int_{x=-3}^{x=3} (9-x^2)^{\frac{1}{2}} dx$$

$= 8 \cdot$ (Area )

$$= 8 \cdot \tfrac{1}{2}(\pi 3^2) = 4 \cdot 9\pi = 36\pi$$

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Change of variables in integration

$$\iint_D f\,dx\,dy = \iint_D f \cdot \left| \det\left(\frac{\partial(x,y)}{\partial(u,v)} \right) \right| du\,dv$$

$$\iiint_D f\,dx\,dy\,dz = \iiint_D f \cdot \left| \det\left(\frac{\partial(x,y,z)}{\partial(u,v,w)} \right) \right| du\,dv\,dw$$

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Polar coordinates

$$x = r\cos\theta \qquad r = (x^2+y^2)^{\frac{1}{2}}$$

$$y = r\sin\theta \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

$$\text{Jacobian} = \det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right) = r$$

Cylindrical coordinates.

$$x = \rho\cos\varphi \qquad \rho = (x^2+y^2)^{\frac{1}{2}}$$

$$y = \rho\sin\varphi \qquad \varphi = \arctan\left(\frac{y}{x}\right)$$

$$z = z \qquad z = z$$

$$\text{Jacobian} = \det\left(\frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)}\right) = \rho$$

Spherical coordinates

$$x = r\sin\theta\cos\varphi \qquad r = (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$y = r\sin\theta\sin\varphi \qquad \varphi = \arctan\left(\frac{y}{x}\right)$$

$$z = r\cos\theta \qquad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \arccos\left(\frac{z}{r}\right)$$

$$\text{Jacobian} = \det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right) = r^2\sin\theta$$