Vector Calculus Lecture 17

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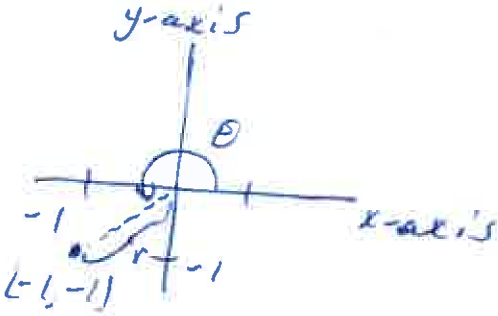
§3.3 Example 1 Convert $(x,y)=(-1,-1)$ to polar coordinates.

Solution

$$r=\sqrt{x^2+y^2}$$
$$\theta=\arctan\left(\frac{y}{x}\right)$$



$$r=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}.$$

$$\theta=\pi+\frac{\pi}{4}=\frac{3\pi}{4} \text{ and indeed } \tan\left(\frac{3\pi}{4}\right)=\frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)}$$
$$=\frac{-\sqrt{2}/2}{-\sqrt{2}/2}=1=\frac{-1}{-1}.$$

So, in polar coordinates, $(x,y)=(-1,-1)$ is
$$(r,\theta)=\left(\sqrt{2},\frac{3\pi}{4}\right).$$

§3.3 Example 2 Convert $(x,y,z)=(1,1,\sqrt{3})$ to cylindrical and spherical coordinates.

Solution:

Cylindrical coordinates:

$$\rho=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{2}$$
$$\varphi=\arctan\left(\frac{y}{x}\right)=\arctan\left(\frac{y}{x}\right)=\arctan\left(\frac{1}{1}\right)=\frac{\pi}{4}$$
$z=z=\sqrt{3}$. So, in cylindrical coordinates
$(x,y,z)=(1,1,\sqrt{3})$ is $(\rho,\varphi,z)=\left(\sqrt{2},\frac{\pi}{4},\sqrt{3}\right)$.

Spherical coordinates

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+1^2+(\sqrt{3})^2} = \sqrt{1+1+3} = \sqrt{5}$$

$$\varphi = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

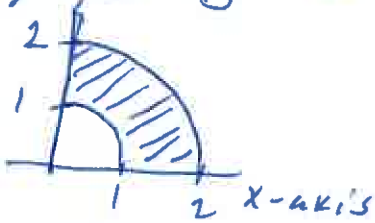
$$\theta = \arccos\left(\frac{z}{r}\right) = \arccos\left(\frac{\sqrt{3}}{\sqrt{5}}\right)$$

So, in spherical coordinates

$(x, y, z) = (1, 1, \sqrt{3})$ is $(r, \varphi, \theta) = \left(\sqrt{5}, \frac{\pi}{4}, \arccos\left(\frac{\sqrt{3}}{\sqrt{5}}\right)\right)$

§3.3 Example 3 Evaluate $\iint_D \log(x^2+y^2)\,dx\,dy$

where D is given by



Solution:

$$\iint_D \log(x^2+y^2)\,dx\,dy = \iint_D \log(x^2+y^2)\, r\,dr\,d\theta$$

$$= \iint_D \log(r^2)\, r\,dr\,d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} \log(r^2)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=2} 2r\log(r)\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} \left(\frac{dr^2}{dr}\log(r) + r^2\frac{d\log r}{dr} - r^2\frac{1}{r}\right) dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} \left(\frac{d(r^2\log r)}{dr} - r\right) dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} r^2 \log r - \frac{r^2}{2} \Big]_{r=1}^{r=2} d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(\left(2^2 \log 2 - \frac{2^2}{2}\right) - \left(1 \cdot \log 1 - \frac{1}{2}\right)\right) d\theta$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(4 \log 2 - 2 - 0 + \frac{1}{2}\right) d\theta$$

$$= \left(4 \log 2 - \frac{3}{2}\right) \int_{\theta=0}^{\theta=\frac{\pi}{2}} d\theta = \left(4 \log 2 - 3/2\right) \theta \Big]_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \left(4 \log 2 - \frac{3}{2}\right) \frac{\pi}{2}$$

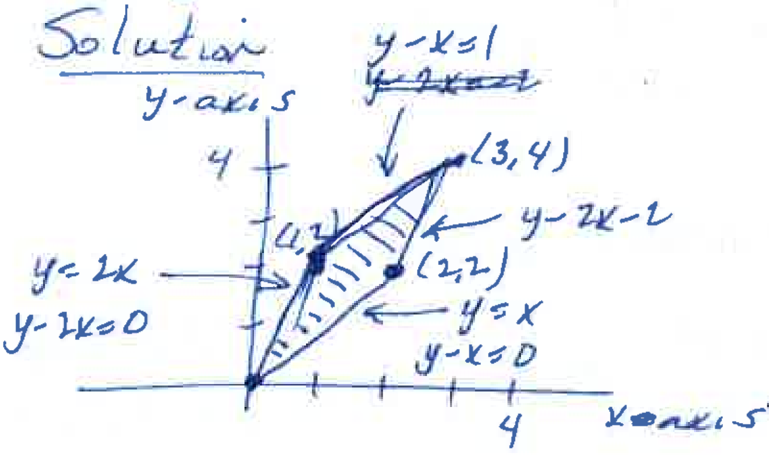
§3.3 Example 4 Let D be the parallelogram with vertices $(0,0)$, $(1,2)$ $(2,2)$, $(3,4)$.

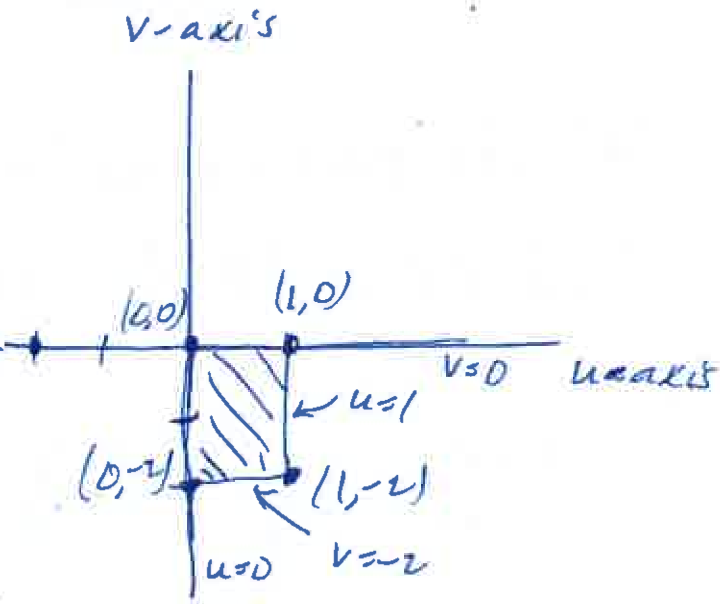
Evaluate

$$\iint_D xy\,dx\,dy$$ by making the

change of variables $x = u - v$, $y = 2u - v$.

Solution





$$y - x = (2u-v) - (u-v) = u$$

$$y - 2x = (2u-v) - 2(u-v) = v.$$

Then

$$\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1-(-2) = 1.$$

So

$$\iint_D xy\,dx\,dy = \iint_D xy \cdot 1\,du\,dv = \iint_D (u-v)(2u-v)\,du\,dv$$

$$= \int_{v=-2}^{v=0} \int_{u=0}^{u=1} (2u^2 - 3uv + v^2)\,du\,dv.$$

$$= \int_{v=-2}^{v=0} 2\frac{u^3}{3} - 3v\frac{u^2}{2} + v^2 u \Big]_{u=0}^{u=1} dv$$

$$= \int_{v=-2}^{v=0} \left(\frac{2}{3} - \frac{3}{2}v + v^2 - (0-0+0)\right) dv$$

$$= \int_{v=-2}^{v=0} \left(\frac{2}{3} - \frac{3}{2}v + v^2\right) dv$$

$$= \frac{2}{3}v - \frac{3}{2}\cdot\frac{v^2}{2} + \frac{v^3}{3}\Big]_{v=-2}^{v=0}$$

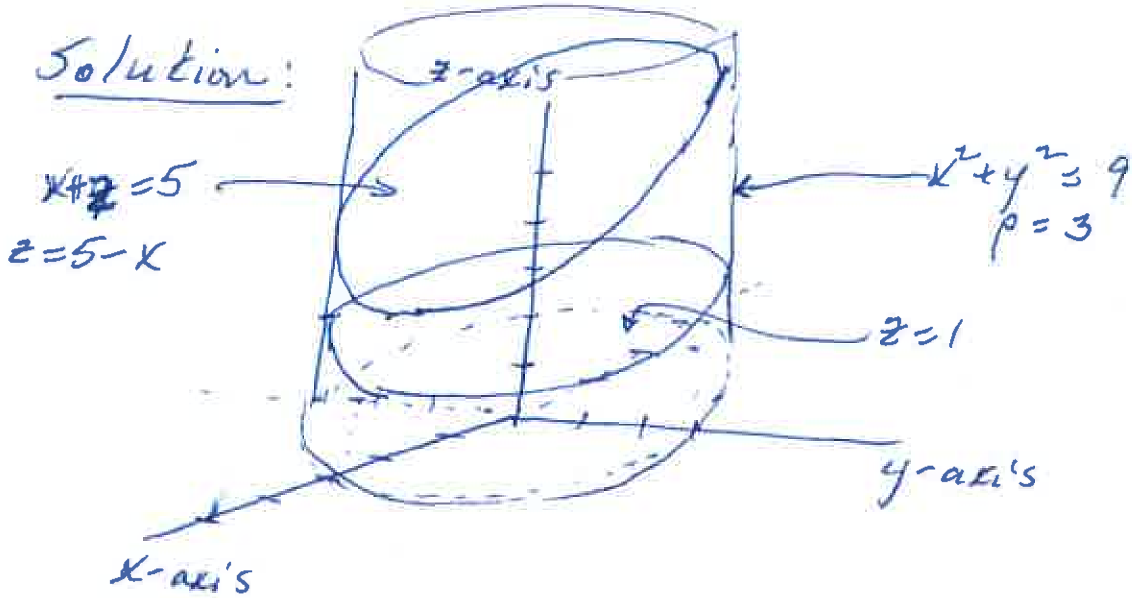
$$= (0-0+0) - \left(\frac{2}{3}(-2) - \frac{3}{4}(-2)^2 + \frac{(-2)^3}{3}\right)$$

$$= \frac{4}{3} + 3 + \frac{8}{3} = 7.$$

§3.3 Example 5 Find the volume of the region bounded by

$$x+z=5, \quad x^2+y^2=9 \text{ and } z=1.$$

Solution:



Use cylindrical coordinates:

$$x=\rho\cos\varphi, \quad y=\rho\sin\varphi, \quad z=z.$$

$$\text{Volume} = \iiint_V dV = \iiint_V \rho \, d\rho \, d\varphi \, dz$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=3} \int_{z=1}^{z=5-\rho\cos\varphi} \rho \, dz \, d\rho \, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=3} \rho z \Big]_{z=1}^{z=5-\rho\cos\varphi} d\rho \, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=3} \left(\rho(5-\rho\cos\varphi)-\rho\right) d\rho \, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=3} (5\rho - \rho^2\cos\varphi - \rho)\, d\rho\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=3} (4\rho - \rho^2\cos\varphi)\, d\rho\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(4\frac{\rho^2}{2} - \frac{\rho^3}{3}\cos\varphi\right)\Big]_{\rho=0}^{\rho=3} d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(\left(\frac{4\cdot 9}{2} - \frac{27}{3}\cos\varphi\right) - (0-0)\right) d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} (18 - 9\cos\varphi)\, d\varphi$$

$$= 18\varphi - 9\sin\varphi\Big]_{\varphi=0}^{\varphi=2\pi}$$

$$= (36\pi - 9\sin 2\pi) - (0 - 9\sin 0)$$

$$= 36\pi.$$