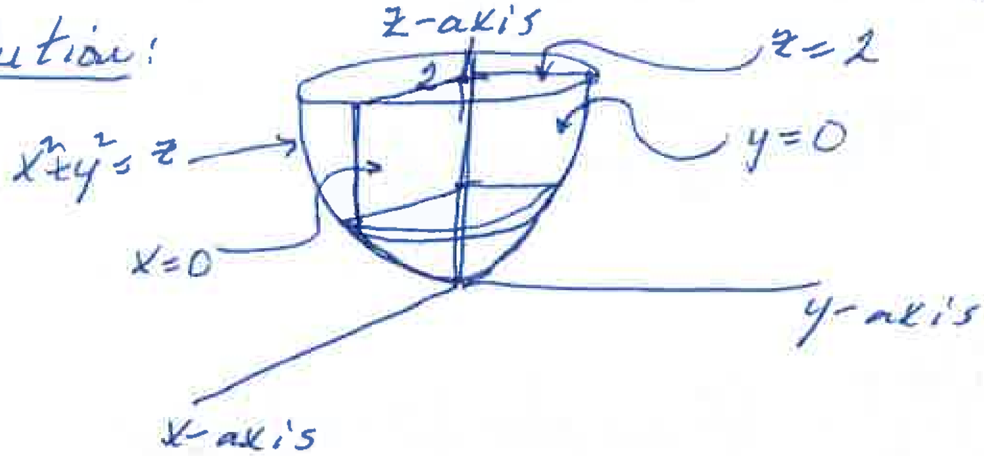
Vector Calculus lecture 18

§3.3 Example 6

Find $\iiint_D x\,dV$ for the region D bounded by

$$z=2,\quad x^2+y^2=z,\quad x\geq 0 \text{ and } y\geq 0$$

Solution:



Use cylindrical coordinates:

$$x=\rho\cos\varphi,\quad y=\rho\sin\varphi,\quad z=z.$$

Then

$$\iiint_D x\,dV=\iiint_D x\,\rho\,d\rho\,d\varphi\,dz=\iiint_D \rho\cos\varphi\cdot\rho\,d\rho\,d\varphi\,dz$$

$$=\int_{z=0}^{z=2}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=\sqrt{z}}\rho^2\cos\varphi\,d\rho\,d\varphi\,dz$$

$$=\int_{z=0}^{z=2}\int_{\varphi=0}^{\varphi=\pi/2}\left.\frac{\rho^3}{3}\cos\varphi\right]_{\rho=0}^{\rho=\sqrt{z}}d\varphi\,dz$$

$$=\int_{z=0}^{z=2}\int_{\varphi=0}^{\varphi=\frac{\pi}{2}}\left(\frac{z^{3/2}}{3}\cos\varphi-0\right)d\varphi\,dz$$

$$= \int_{z=0}^{z=2} \frac{1}{3} z^{3/2} \sin\varphi \Big]_{\varphi=0}^{\varphi=\pi/2} dz$$

$$= \int_{z=0}^{z=2} \left(\frac{1}{3} z^{3/2} \sin\frac{\pi}{2} - \frac{1}{3} z^{3/2} \sin 0\right) dz$$

$$= \int_{z=0}^{z=2} \frac{1}{3} z^{3/2} dz = \frac{1}{3}\cdot\frac{2}{5} z^{5/2}\Big]_{z=0}^{z=2}$$

$$= \frac{2}{3\cdot 5} 2^{5/2} - 0 = \frac{8\sqrt{2}}{15}.$$

§3.3 Example 7 If D is the unit sphere centred at $(0,0,0)$ evaluate

$$\iiint_D \exp\left((x^2+y^2+z^2)^{3/2}\right) dV.$$

Solution: Use spherical coordinates.

$$r = \sqrt{x^2+y^2+z^2} \quad \text{and} \quad \det\left(\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}\right) = r^2 \sin\theta$$

So

$$\iiint_D \exp\left((x^2+y^2+z^2)^{3/2}\right) dV$$

$$= \iiint_D \exp(r^3)\, r^2 \sin\theta \, dr\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=1} r^2 e^{r^3} \sin\theta \, dr\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \sin\theta \left. \tfrac{1}{3} e^{r^3} \right]_{r=0}^{r=1} d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} \sin\theta \left(\tfrac{1}{3} e^1 - \tfrac{1}{3} e^0\right) d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sin\theta \left(\tfrac{1}{3} e - \tfrac{1}{3}\right) \varphi \Big]_{\varphi=0}^{\varphi=2\pi} d\theta$$

$$= \int_{\theta=0}^{\theta=\pi} \sin\theta \ \tfrac{1}{3}(e-1)(2\pi-0) \, d\theta$$

$$= \tfrac{1}{3}(e-1)\, 2\pi \, (-\cos\theta)\Big]_{\theta=0}^{\theta=\pi}$$

$$= \frac{2\pi}{3}(e-1)\big(-\cos\pi - (-\cos 0)\big)$$

$$= \frac{2\pi}{3}(e-1)\big(-(-1)+1\big)$$

$$= \frac{4\pi}{3}(e-1).$$