Vector calculus lecture 19

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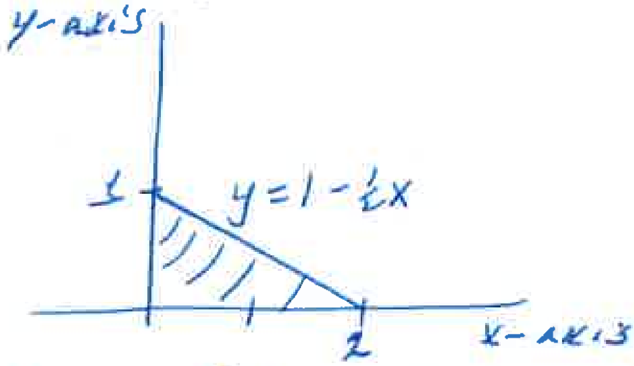
UniHelp

§3.4 Example 1: The thickness of a triangular plate varies as

$$f(x,y) = (1+xy)\,\text{cm}$$

Find the average thickness of the plate



Solution: Area of triangle $= \frac{1}{2}(1\cdot 2) = \frac{1}{2}\cdot 2 = 1$

$$\text{Average thickness} = \frac{\iint_D (1+xy)\,dx\,dy}{\text{Area of } D} = \frac{\iint_D (1+xy)\,dx\,dy}{1}$$

$$= \int_{x=0}^{x=2}\int_{y=0}^{y=1-\frac{1}{2}x} (1+xy)\,dy\,dx = \int_{x=0}^{x=2} \left. y + x\frac{y^2}{2}\right]_{y=0}^{y=1-\frac{1}{2}x} dx$$

$$= \int_{x=0}^{x=2} \left((1-\tfrac{1}{2}x) + \frac{x}{2}(1-\tfrac{1}{2}x)^2 \right) dx$$

$$= \int_{x=0}^{x=2} \left(1 - \tfrac{1}{2}x + \frac{x}{2}(1 - x + \tfrac{1}{4}x^2)\right) dx$$

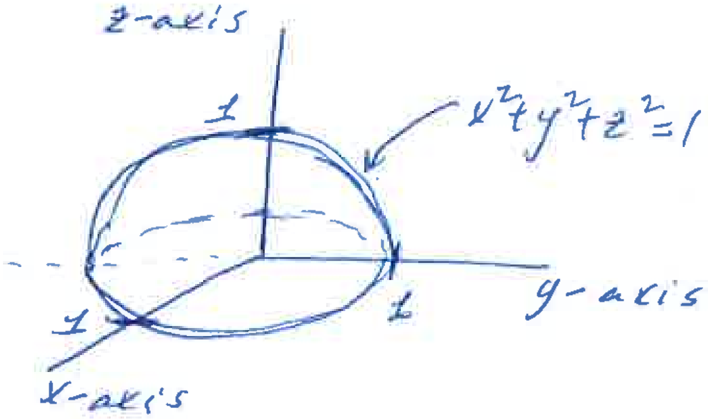
$$= \int_{x=0}^{x=2} \left(1 - \tfrac{1}{2}x + \frac{x}{2} - \frac{x^2}{2} + \frac{1}{8}x^3\right) dx$$

$$= \left. x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{8}\frac{x^4}{4}\right]_{x=0}^{x=2} = 2 - \frac{1}{6}\cdot 8 + \frac{1}{8\cdot 4}\cdot 2^4 - (0-0+0)$$

$$= 2 - \frac{4}{3} + \frac{1}{2} = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}\,\text{cm}.$$

§3.4 Example 2 Find the centre of mass of a solid hemisphere of radius 1 centred at $(0,0,0)$ for $z \geq 0$, if the mass density per unit volume μ is constant.

Solution



By symmetry, the

$$\begin{pmatrix}\text{x-coordinate}\\ \text{of centre of mass}\end{pmatrix} = \begin{pmatrix}\text{y-coordinate}\\ \text{of centre of mass}\end{pmatrix} = 0.$$

Then

$$\begin{pmatrix}\text{z-coordinate}\\ \text{of centre of mass}\end{pmatrix} = \frac{\iiint_D z\mu\, dV}{\iiint_D \mu\, dV} = \frac{\mu\iiint_D z\, dV}{\mu\iiint_D dV}$$

$$= \frac{\iiint_D z\, dV}{\begin{pmatrix}\text{volume of}\\ \text{hemisphere}\end{pmatrix}} = \frac{1}{\frac{1}{2}\left(\frac{4}{3}\pi\cdot 1^3\right)}\iiint_D z\, dV$$

since μ is a constant.

So

$$\text{z-coordinate of centre of mass} = \frac{3}{2\pi}\iiint_D z\,\rho\,dz\,d\rho\,d\varphi$$

$$= \frac{3}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=0}^{z=\sqrt{1-\rho^2}} \rho z\,dz\,d\rho\,d\varphi$$

$$= \frac{3}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=1} \rho\frac{z^2}{2}\Big]_{z=0}^{z=\sqrt{1-\rho^2}} d\rho\,d\varphi$$

$$= \frac{3}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=1}\left(\rho\frac{(1-\rho^2)}{2} - 0\right)d\rho\,d\varphi$$

$$= \frac{3}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{\rho=0}^{\rho=1}\frac{1}{2}(\rho-\rho^3)\,d\rho\,d\varphi$$

$$= \frac{3}{2\pi}\int_{\varphi=0}^{\varphi=2\pi}\frac{1}{2}\left(\frac{\rho^2}{2}-\frac{\rho^4}{4}\right)\Big]_{\rho=0}^{\rho=1} d\varphi$$

$$= \frac{3}{4\pi}\int_{\varphi=0}^{\varphi=2\pi}\left(\left(\frac{1}{2}-\frac{1}{4}\right)-(0-0)\right)d\varphi$$

$$= \frac{3}{4\pi}\cdot\frac{1}{4}\varphi\Big]_{\varphi=0}^{\varphi=2\pi} = \frac{3}{16\pi}(2\pi-0) = \cancel{\frac{3\pi}{8}}\ \frac{3}{8}.$$

§3.4 Example 3 If the density of material inside the cone

$$z = 2\sqrt{x^2+y^2} \text{ with } z \le 4$$

varies as

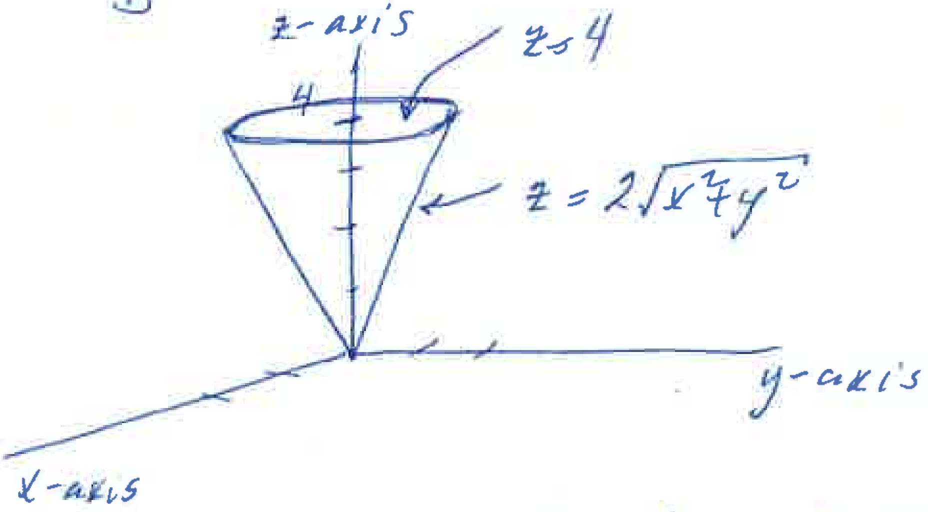
$$\mu = 5 - z,$$

find the moment of inertia about the z-axis.

Solution I_z = moment of inertia about z-axis

= weighted distance from z-axis

$$= \iiint_D (x^2+y^2)\,\mu\, dV$$



Use cylindrical coordinates ρ, φ, z.

$$\iiint_D (x^2+y^2)\mu\, dV = \iiint_D (x^2+y^2)(5-z)\, dz\, d\rho\, d\varphi \cdot \rho$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=2} \int_{z=2\rho}^{z=4} \rho^2 (5-z)\, \rho\, dz\, d\rho\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=2} \rho^3\left(5z - \frac{z^2}{2}\right)\Big]_{z=2\rho}^{z=4} d\rho\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=2} \left(\rho^3(20-8) - \rho^3\left(10\rho - \frac{4\rho^2}{2}\right)\right) d\rho\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\rho=0}^{\rho=2} (12\rho^3 - 10\rho^4 + 2\rho^5)\, d\rho\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} 12\frac{\rho^4}{4} - 10\frac{\rho^5}{5} + 2\frac{\rho^6}{6}\Big]_{\rho=0}^{\rho=2} d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \left(\left(12\cdot\frac{16}{4} - \frac{10}{5}\cdot 32 + \frac{2}{6}\cdot 64\right) - (0-0+0)\right) d\varphi$$

$$= \left(48 - 64 + \frac{64}{3}\right)\varphi\Big]_{\varphi=0}^{\varphi=2\pi}$$

$$= \left(-16 + 21 + \frac{1}{3}\right)\cdot 2\pi = \frac{16}{3}\cdot 2\pi = \frac{32\pi}{3}.$$