Vector Calculus Lecture 20

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A. Ram
UniMelb.

§4.1 Example 1 Let

$$\vec{c}(t) = (\cos t, \sin t, t) \quad \text{for} \quad 0 \le t \le 2\pi$$

Evaluate $\int_{\vec{c}} (xy+z)\, ds$.

Solution

$$\int_{\vec{c}} (xy+z)\, ds = \int_{\vec{c}} (xy+z) \frac{ds}{dt}\, dt = \int_{\vec{c}} (xy+z) \left|\frac{d\vec{c}}{dt}\right| dt$$

$\frac{d\vec{c}}{dt} = (-\sin t, \cos t, 1)$ and

$$\left|\frac{d\vec{c}}{dt}\right| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}.$$

So

$$\int_{\vec{c}} (xy+z)\, ds = \int_{\vec{c}} (xy+z)\sqrt{2}\, dt$$

$$= \int_{t=0}^{t=2\pi} \sqrt{2}(\cos t \sin t + t)\, dt = \sqrt{2}\left(\frac{\sin^2 t}{2} + \frac{t^2}{2}\right)\Bigg]_{t=0}^{t=2\pi}$$

$$= \sqrt{2}\left(\frac{\sin^2 2\pi}{2} + \frac{(2\pi)^2}{2}\right) - \sqrt{2}\left(\frac{\sin^2 0}{2} + \frac{0^2}{2}\right)$$

$$= \sqrt{2}(0 + 2\pi^2) - \sqrt{2}(0+0)$$

$$= 2\sqrt{2}\pi^2.$$

§4.1 Example 2 Let $f(x,y) = y - x$ and

$$\vec{c}(t) = \begin{cases} (2t, t), & \text{for } 0 \le t \le 1 \\ (t+1, 5-4t), & \text{for } 1 \le t \le 3. \end{cases}$$

Evaluate $\int_{\vec{c}} f\, ds$.

Solution:

$$\frac{d\vec{c}}{dt} = \begin{cases} (2,1), & \text{for } 0 \le t \le 1, \\ (1,-4), & \text{for } 1 \le t \le 3. \end{cases}$$

$$\left|\frac{d\vec{c}}{dt}\right| = \begin{cases} \sqrt{2^2+1^2} = \sqrt{5}, & \text{for } 0 \le t \le 1, \\ \sqrt{1^2+4^2} = \sqrt{17}, & \text{for } 1 \le t \le 3. \end{cases}$$

So

$$\int_{\vec{c}} f\, ds = \int_{t=0}^{t=1} f \frac{ds}{dt}\, dt + \int_{t=1}^{t=3} f \frac{ds}{dt}\, dt$$

$$= \int_{t=0}^{t=1} (y-x)\sqrt{5}\, dt + \int_{t=1}^{t=3} (y-x)\sqrt{17}\, dt$$

$$= \int_{t=0}^{t=1} (t-2t)\sqrt{5}\, dt + \int_{t=1}^{t=3} (5-4t-(t+1))\sqrt{17}\, dt$$

$$= \int_{t=0}^{t=1} -\sqrt{5}\, t\, dt + \int_{t=1}^{t=3} (4-5t)\sqrt{17}\, dt$$

$$= -\sqrt{5}\frac{t^2}{2}\Big]_{t=0}^{t=1} + \left(4t - \frac{5t^2}{2}\right)\sqrt{17}\Big]_{t=1}^{t=3}$$

$$= -\sqrt{5}\left(\tfrac{1}{2}-0\right) + \sqrt{17}\left(\left(4\cdot 3 - \tfrac{5\cdot 9}{2}\right) - \left(4 - \tfrac{5}{2}\right)\right)$$

$$= -\frac{\sqrt{5}}{2} + \sqrt{17}\left(12 - \frac{45}{2} - 4 + \frac{5}{2}\right)$$

$$= -\tfrac{1}{2}\sqrt{5} + \sqrt{17}\left(8 - \tfrac{40}{2}\right) = -\tfrac{1}{2}\sqrt{5} + \sqrt{17}(8-20)$$

$$= -\tfrac{1}{2}\sqrt{5} - 12\sqrt{17}.$$

§4.1 Example 3ai Find a parametrization of $y=z^2$, $x=2$ from $(2,0,0)$ to $(2,9,3)$.

Solution $c(t) = (2, t^2, t)$ for $0 \le t \le 3$ has $x=2$, has $y=z^2$, begins at $(2, 0^2, 0)$, and ends at $(2, 3^2, 3)$.

§4.1 Example 3aii Find a parametrization of $y=z^2$, $x=2$ from $(2,9,3)$ to $(2,0,0)$

Solution: $c(t) = (2, (3-t)^2, (3-t))$ for $0 \le t \le 3$ has $x=2$, has $y=z^2$, begins at $(2, (3-0)^2, 3-0)$ and ends at $(2, (3-3)^2, 3-0)$. //

§4.1 Example 3b Find a parametrization for the line joining $(1,0,2)$ to $(4,3,1)$.

Solution:

$$\vec{r}(t) = (1,0,2) + t(4-1,\, 3-0,\, 1-2)$$

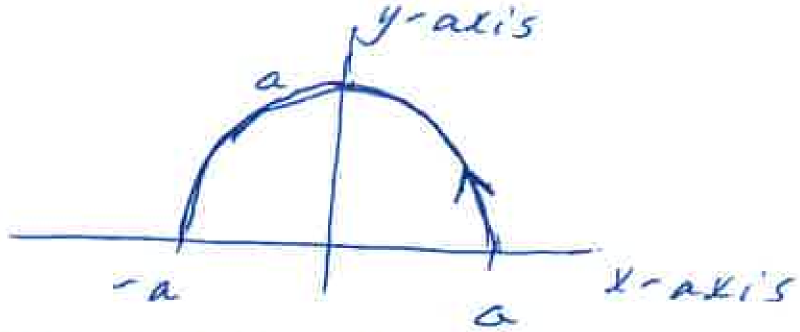
$$= (1,0,2) + t(3,3,-1)$$

$$= (1+3t,\, 0+3t,\, 2-t) = (1+3t,\, 3t,\, 2-t)$$

for $0 \le t \le 1$, is a line which begins at $(1,0,2)$ and ends at $(1+3,\, 3,\, 2-1) = (4,3,1)$.

§4.1 Example 3ci Find a parametrization for the ~~line~~ semicircle $y = \sqrt{a^2 - x^2}$ oriented anticlockwise.

Solution



$$\vec{r}(t) = \boxed{(a\cos t,\, a\sin t)} \text{ for } 0 \le t \le \pi$$

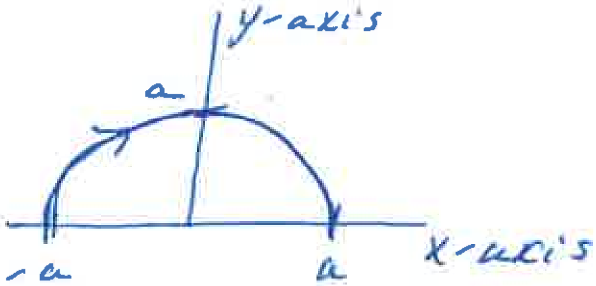
has $y = a\sin t = \sqrt{a^2 - a^2\cos^2 t} = \sqrt{a^2 - x^2}$,

begins at $(a\cos 0,\, a\sin 0) = (a, 0)$

and ends at $(a\cos\pi,\, a\sin\pi) = (-a, 0)$.

§4.1 Example c ii Find a parametrisation for the semicircle $y=\sqrt{a^2-x^2}$ oriented clockwise.

Solution:



$\vec{c}(t) = (a\cos(\pi-t), a\sin(\pi-t))$ for $0 \le t \le \pi$

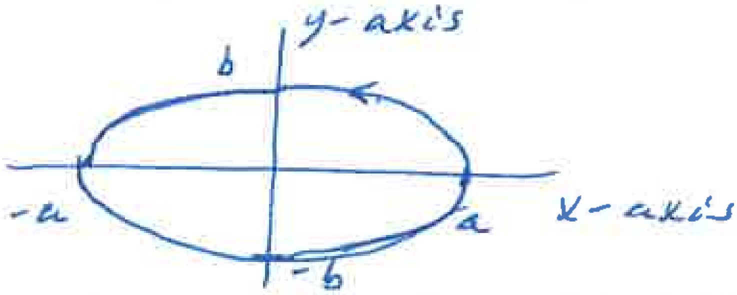
has $y = a\sin(\pi-t) = \sqrt{a^2 - a^2\cos^2(\pi-t)} = \sqrt{a^2-x^2}$

begins at $(a\cos\pi, a\sin\pi) = (-a, 0)$

and ends at $(a\cos 0, a\sin 0) = (a, 0)$.

§4.1 Example d i Find a parametrization for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ traversed anticlockwise.

Solution:



$\vec{c}(t) = (a\cos t, b\sin t)$ for $0 \le t < 2\pi$

has $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1$;

at $t=0$ is $(a,0)$, at $t=\frac{\pi}{2}$ is $(0,b)$, at $t=\pi$ is $(-a,0)$ at $t=\frac{3\pi}{2}$ is $(0,-b)$.

§4.1 Example dii Find a parametrisation for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ traversed clockwise.

Solution $\vec{c}(t)=(a\cos(2\pi-t), b\sin(2\pi-t))$ for $0\le t\le 2\pi$, has

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a^2\cos^2(2\pi-t)}{a^2}+\frac{b^2\sin^2(2\pi-t)}{b^2}=1;$$

at $t=0$ is $(a,0)$, at $t=\frac{\pi}{2}$ is $(0,-b)$, at $t=\pi$ is $(-a,0)$, at $t=\frac{3\pi}{2}$ is $(b,0)$.