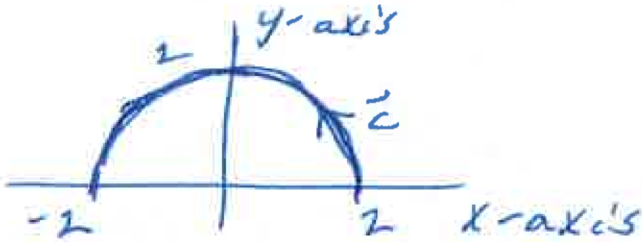
§4.2 Example 1 Determine the work done by the force

$$\vec{F}(x,y) = (-y, 0)$$

to move a particle around the semicircle $y=\sqrt{4-x^2}$ from $(2,0)$ to $(-2,0)$.

Solution:



Compute $\int_{\vec{c}} \vec{F}\cdot d\vec{s}$ for

$$\vec{c}(t) = (2\cos t, 2\sin t) \text{ for } 0 \le t \le \pi.$$

Since

$$\frac{d\vec{c}}{dt} = \frac{d\vec{s}}{dt} = (-2\sin t, 2\cos t), \text{ then}$$

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_{\vec{c}} \vec{F}\cdot \frac{d\vec{s}}{dt}\,dt = \int_{\vec{c}} \vec{F}\cdot\frac{d\vec{c}}{dt}\,dt$$

$$= \int_{\vec{c}} (-y\hat{\imath}+0\hat{\jmath})\cdot(-2\sin t\,\hat{\imath}+2\cos t\,\hat{\jmath})\,dt$$

$$= \int_{\vec{c}} (-2\sin t\,\hat{\imath}+0\hat{\jmath})\cdot(-2\sin t\,\hat{\imath}+2\cos t\,\hat{\jmath})\,dt$$

$$= \int_{\vec{c}} 4\sin^2 t\,dt.$$

$$= \int_{t=0}^{t=\pi} 2(1-\cos 2t)\,dt, \quad \text{since } \cos 2t = \cos^2 t - \sin^2 t$$
$$= 1 - \sin^2 t - \sin^2 t$$
$$= 1 - 2\sin^2 t$$

$$= 2\left(t - \frac{\sin 2t}{2}\right)\Big|_{t=0}^{t=\pi}$$

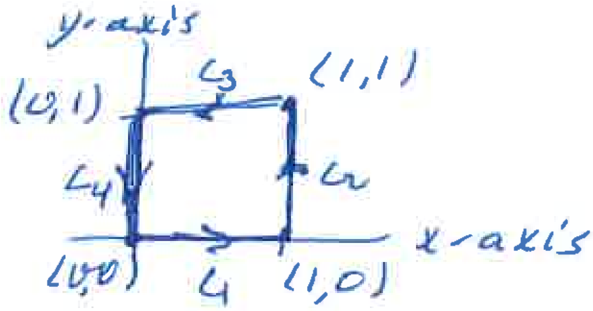
$$= 2\left(\pi - \frac{\sin 2\pi}{2} - \left(0 - \frac{\sin 0}{2}\right)\right)$$

$$= 2(\pi - 0 - 0 + 0) = 2\pi.$$

<u>§4.2 Example 2</u> Let $\vec{c}$ be the perimeter of the unit square with vertices at $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, oriented anticlockwise. Evaluate

$$\int_c x^2\,dx + xy\,dy.$$

<u>Solution:</u>



$c = c_1 \cup c_2 \cup c_3 \cup c_4$

$\vec{c}_1(t) = (t, 0)$ for $0 \le t \le 1$

$\vec{c}_2(t) = (1, t)$ for $0 \le t \le 1$

$\vec{c}_3(t) = (1-t, 1)$ for $0 \le t \le 1$

$\vec{c}_4(t) = (0, 1-t)$ for $0 \le t \le 1$.

Then

$$\int_c x^2\,dx + xy\,dy = \int_{c_1} x^2\,dx + xy\,dy + \int_{c_2} x^2\,dx + xy\,dy + \int_{c_3} x^2\,dx + xy\,dy + \int_{c_4} x^2\,dx + xy\,dy$$

$$= \int_{c_1}\left(x^2\frac{dx}{dt} + xy\frac{dy}{dt}\right)dt + \int_{c_2}\left(x^2\frac{dx}{dt} + xy\frac{dy}{dt}\right)dt$$

$$+ \int_{c_3}\left(x^2\frac{dx}{dt} + xy\frac{dy}{dt}\right)dt + \int_{c_4}\left(x^2\frac{dx}{dt} + xy\frac{dy}{dt}\right)dt$$

$$= \int_{t=0}^{t=1} (t^2 \cdot 1 + t \cdot 0 \cdot 0)\, dt + \int_{t=0}^{t=1} (1^2 \cdot 0 + 1 \cdot t \cdot 1)\, dt$$

$$+ \int_{t=0}^{t=1} ((1-t)^2(-1) + (1-t) \cdot t \cdot 0)\, dt$$

$$+ \int_{t=0}^{t=1} (0^2 \cdot 0 + 0 \cdot (1-t)(-1))\, dt$$

$$= \frac{t^3}{3}\Big]_{t=0}^{t=1} + \frac{t^2}{2}\Big]_{t=0}^{t=1} + \int_{t=0}^{t=1} -(1-2t+t^2)\, dt + 0.$$

$$= \frac{1}{3} + \frac{1}{2} + -\left(t - t^2 + \frac{t^3}{3}\right)\Big]_{t=0}^{t=1}$$

$$= \frac{5}{6} + \left(-(1-1+\tfrac{1}{3}) - (-(0-0+0))\right) = \frac{5}{6} - \frac{1}{3}$$

$$= \frac{3}{6} = \frac{1}{2}$$