Vector Calculus Lecture 24

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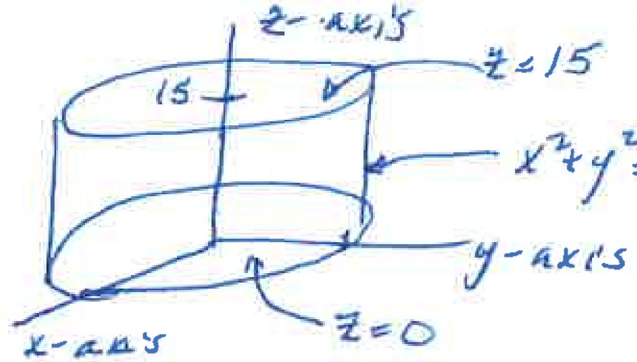
§4.4 Example 2 Let S be the closed cylinder with

$$x^2+y^2=9, \quad z=0 \quad \text{and} \quad z=15.$$

Find $\iint_S z\,dS$.

Solution



The top surface is parametrised by

$$\Phi_1(x,y)=(x,y,15) \quad \text{for} \quad x^2+y^2\le 9.$$

This is of the form $\Phi(x,y)=(x,y,f(x,y))$ and so

$$|\vec{T}_x\times\vec{T}_y|=\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}=\sqrt{0^2+0^2+1^2}=1$$

and

$$\iint_{S_1} z\,dS=\iint_{S_1} z\,|\vec{T}_x\times\vec{T}_y|\,dx\,dy=\iint_{S_1}15\cdot 1\,dx\,dy$$

$$=15\iint_{S_1}dx\,dy=15\left(\begin{array}{c}\text{area of circle}\\ \text{of radius } 3\end{array}\right)=15\cdot\pi\cdot 3^2$$

For the bottom surface $z=0$ so

$$\iint_{S_2} z\,dS=\iint_{S_2}0\cdot dS=0.$$

The side of the cylinder is the surface S_3 parametrised by

$$\Phi(x,z) = (x, \sqrt{9-x^2}, z) \quad \text{for} \quad \begin{array}{l} -3 \le x \le 3 \\ 0 \le z \le 15 \end{array}$$

Alternatively in cylindrical coordinates this surface is $\rho = 3$ so that

$$\Phi(\varphi, z) = (3\cos\varphi, 3\sin\varphi, z) \quad \begin{array}{l} 0 \le \varphi \le 2\pi \\ 0 \le z \le 15. \end{array}$$

So

$$\vec{T}_\varphi = \left(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\right) = (-3\sin\varphi, 3\cos\varphi, 0)$$

$$\vec{T}_z = \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (0,0,1)$$

$$\vec{T}_\varphi \times T_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\varphi & 3\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{array}{l} \hat{i}(3\cos\varphi - 0) \\ -\hat{j}(-3\sin\varphi - 0) \\ +\hat{k}\cdot 0 \end{array}$$

$$= (3\cos\varphi, 3\sin\varphi, 0), \quad \text{and}$$

$$|\vec{T}_\varphi \times \vec{T}_z| = \sqrt{3^2\cos^2\varphi + 3^2\sin^2\varphi} = 3.$$

So

$$\iint_{S_3} z\,dS = \iint_{S_3} z\,|\vec{T}_\varphi \times \vec{T}_z|\,d\varphi\,dz = \iint_{S_3} z\cdot 3\,d\varphi\,dz$$

$$= \int_{z=0}^{z=15}\int_{\varphi=0}^{\varphi=2\pi} 3z\,d\varphi\,dz = 2\pi\int_{z=0}^{z=15} 3z\,dz = 2\pi\cdot 3\frac{z^2}{2}\Big]_{z=0}^{z=15}$$

$$= 3\pi\cdot 15^2.$$

All together

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$

$$= 3^2\pi \cdot 15 + 0 + 3\pi \cdot 15^2$$

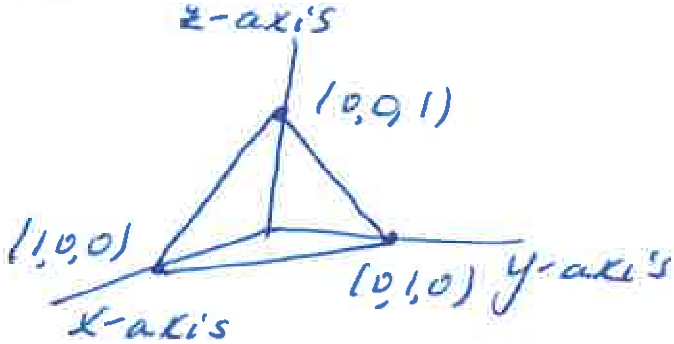
$$= 3\pi \cdot 15(3+15) = 3\cdot 3\cdot 5\pi \cdot 3 \cdot 6 = 3^4 \cdot 10\pi.$$

$$= 810\pi.$$

<u>§4.4 Example 3</u> Let S be the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Find

$$\iint_S x\, dS$$

<u>Solution</u>



Let

$$\vec{v}_1 = (0,1,0)-(1,0,0) = (-1,1,0)$$
$$\vec{v}_2 = (0,0,1)-(1,0,0) = (-1,0,1).$$

Then $\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{matrix} \hat{i}(1-0) \\ -\hat{j}(-1-0) \\ +k(0-(-1)) \end{matrix} = \hat{i}+\hat{j}+\hat{k} = (1,1,1)$

is a normal vector to the plane

$$(x-1, y-0, z-0)\cdot(1,1,1) = 0$$

which is $x+y+z=1$. The surface is parametrized by

$$\Phi(x,y) = (x, y, 1-x-y) \text{ with } \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{matrix}$$

So $\vec{T}_x = (1,0,-1)$, $\vec{T}_y = (0,1,-1)$ and

$$\vec{T}_x \times \vec{T}_y = \hat{i}+\hat{j}+\hat{k} \quad \text{and} \quad |\vec{T}_x \times \vec{T}_y| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

So
$$\iint_S x\,dS = \iint_S x\,|\vec{T}_x \times \vec{T}_y|\,dx\,dy = \iint_S \sqrt{3}\,x\,dx\,dy$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \sqrt{3}\,x\,dy\,dx = \int_{x=0}^{x=1} \sqrt{3}\,xy\Big]_{y=0}^{y=1-x} dx$$

$$= \int_{x=0}^{x=1} \sqrt{3}\,x(1-x)\,dx = \int_{x=0}^{x=1} \sqrt{3}(x-x^2)\,dx$$

$$= \sqrt{3}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big]_{x=0}^{x=1} = \sqrt{3}\left(\frac{1}{2}-\frac{1}{3}-(0-0)\right)$$

$$= \sqrt{3}\cdot\frac{1}{6} = \frac{\sqrt{3}}{6}.$$