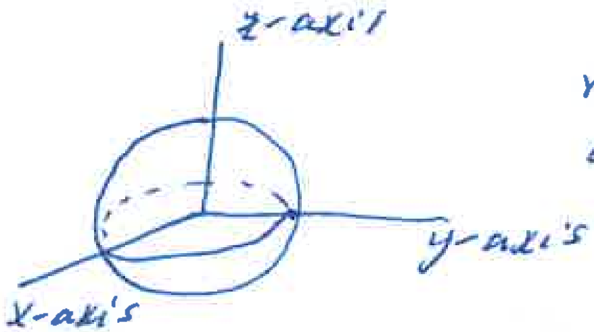
Vector Calculus Lecture 25

§4.5 Example 1 Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where

$$\vec{F} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

and S is the unit sphere centred at $(0,0,0)$.

Solution:



$r=1$ in spherical coordinates.

$$\Phi(\varphi,\theta)=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

with $\quad 0\le\varphi\le 2\pi$ and $0\le\theta\le\pi$.

Then

$$\vec{T}_\varphi=\left(\frac{\partial x}{\partial\varphi},\frac{\partial y}{\partial\varphi},\frac{\partial z}{\partial\varphi}\right)=(\sin\theta(-\sin\varphi),\ \sin\theta\cos\varphi,\ 0)$$

$$\vec{T}_\theta=\left(\frac{\partial x}{\partial\theta},\frac{\partial y}{\partial\theta},\frac{\partial z}{\partial\theta}\right)=(\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)$$

and

$$d\vec{S}=\vec{T}_\varphi\times\vec{T}_\theta=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0\\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta\end{vmatrix}$$

$$\begin{aligned}=\ &\hat{\imath}(-\sin^2\theta\cos\varphi-0)\\ &-\hat{\jmath}(\sin^2\theta\sin\varphi-0)\\ &+\hat{k}(-\sin\theta\cos\theta\sin^2\varphi-\sin\theta\cos\theta\cos^2\varphi)\end{aligned}$$

$$=(-\sin^2\theta\cos\varphi,\ -\sin^2\theta\sin\varphi,\ -\sin\theta\cos\theta).$$

So

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot(\vec{T}_\varphi \times \vec{T}_\theta)\, d\theta\, d\varphi$$

$$= \iint_S (x, y, z)\cdot(-\sin^2\theta\cos\varphi, -\sin^2\theta\sin\varphi, -\sin\theta\cos\theta)\, d\theta\, d\varphi$$

$$= \iint_S \left(-x\sin^2\theta\cos\varphi - y\sin^2\theta\sin\varphi - z\sin\theta\cos\theta\right) d\theta\, d\varphi$$

$$= \iint_S \begin{pmatrix} -\sin\theta\cos\varphi\sin^2\theta\cos\varphi \\ -\sin\theta\sin\varphi\sin^2\theta\sin\varphi \\ -\cos\theta\sin\theta\cos\theta \end{pmatrix} d\theta\, d\varphi$$

$$= \iint_S \left(-\sin^3\theta\cos^2\varphi - \sin^3\theta\sin^2\varphi - \sin\theta\cos^2\theta\right) d\theta\, d\varphi$$

$$= \iint_S \left(-\sin^3\theta - \sin\theta\cos^2\theta\right) d\theta\, d\varphi$$

$$= \iint_S -\sin\theta\left(\sin^2\theta + \cos^2\theta\right) d\theta\, d\varphi$$

$$= \iint_S -\sin\theta\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} -\sin\theta\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi} \cos\theta\Big]_{\theta=0}^{\theta=\pi} d\varphi = \int_{\varphi=0}^{\varphi=2\pi} (-1-1)\, d\varphi = -2\varphi\Big]_{\varphi=0}^{\varphi=2\pi} = -4\pi.$$

Vect. Calc. Lect. 25 18.09.2018 UniMelb A. Ram (3)

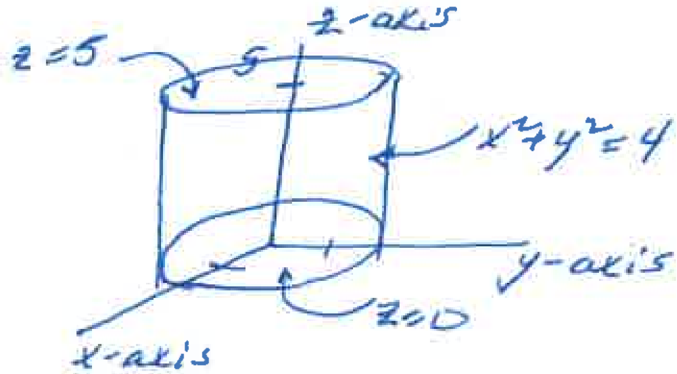
<u>§4.5 Example 2</u> Evaluate

$$\iint_S (x^3\hat{i} + y^3\hat{j}) \cdot d\vec{S}$$

where S is the closed cylinder

$$x^2+y^2=4, \quad z=0, \quad z=5.$$

<u>Solution:</u>



In cylindrical coordinates S_3, the sides of the cylinder with $\rho=2$ is

$$\Phi_3(\varphi, z) = (2\cos\varphi, 2\sin\varphi, z) \text{ with } \begin{matrix} 0\le\varphi\le 2\pi \\ 0\le z\le 5 \end{matrix}$$

The bottom of the cylinder S_1 has

$$\Phi_1(r,\varphi) = (r\cos\varphi, r\sin\varphi, 0) \text{ with } \begin{matrix} 0\le r\le 2 \\ 0\le\varphi\le 2\pi \end{matrix}$$

The top of the cylinder S_2 has

$$\Phi_2(r,\varphi) = (r\cos\varphi, r\sin\varphi, 5) \text{ with } \begin{matrix} 0\le r\le 2 \\ 0\le\varphi\le 2\pi \end{matrix}$$

For the surface S_1,

$d\vec{S} = \vec{T}_r \times \vec{T}_\varphi = -|\vec{T}_r\times\vec{T}_\varphi|\,\hat{k}$ and

$$\iint_{S_1} (x^3\hat{i}+y^3\hat{j})\cdot d\vec{S} = \iint_{S_1} (x^3\hat{i}+y^3\hat{j})\cdot(-\hat{k})\,|\vec{T}_r\times\vec{T}_\varphi|\,d\cancel{S}\,d\varphi$$

$$= \iint_{S_1} 0\,|\vec{T}_r\times\vec{T}_\varphi|\,dS = 0.$$

For the surface S_2

$$d\vec{S} = \vec{T}_r \times \vec{T}_\varphi = |\vec{T}_r \times \vec{T}_\varphi| \hat{k} \quad \text{and}$$

$$\iint_{S_2} (x^3\hat{\imath} + y^3\hat{\jmath}) \cdot d\vec{S} = \iint_{S_2} (x^3\hat{\imath} + y^3\hat{\jmath}) \cdot \hat{k}\, |\vec{T}_r \times \vec{T}_\varphi|\, dr\, d\varphi$$

$$= \iint_{S_2} 0\, dS = 0.$$

For the surface S_3:

$$\vec{T}_\varphi = \left(\frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}\right) = (-2\sin\varphi, 2\cos\varphi, 0)$$

$$\vec{T}_z = \left(\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}\right) = (0, 0, 1)$$

$$\vec{T}_\varphi \times \vec{T}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin\varphi & 2\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{array}{l} \hat{\imath}(2\cos\varphi - 0) \\ -\hat{\jmath}(-2\sin\varphi - 0) = (2\cos\varphi, 2\sin\varphi, 0) \\ +\hat{k}(0-0) \end{array}$$

which is outward pointing. So

$$\iint_{S_3} (x^3\hat{\imath} + y^3\hat{\jmath}) \cdot d\vec{S} = \iint_{S_3} (x^3\hat{\imath} + y^3\hat{\jmath}) \cdot (\vec{T}_\varphi \times \vec{T}_z)\, d\varphi\, dz$$

$$= \iint_{S_3} (x^3\hat{\imath} + y^3\hat{\jmath}) \cdot (2\cos\varphi\, \hat{\imath} + 2\sin\varphi\, \hat{\jmath})\, d\varphi\, dz$$

$$= \iint_{S_3} (2x^3\cos\varphi + 2y^3\sin\varphi)\, d\varphi\, dz$$

$$= \iint_{S_3} \left(2(2\cos\varphi)^3\cos\varphi + 2(2\sin\varphi)^3\sin\varphi\right) d\varphi\, dz$$

$$= 4^2\int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}(\cos^4\varphi+\sin^4\varphi)\,d\varphi\,dz$$

$$= 4^2\int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}(\cos^4\varphi+(1-\cos^2\varphi)^2)\,d\varphi\,dz$$

$$= 4^2\int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}(1-2\cos^2\varphi+2\cos^4\varphi)\,d\varphi\,dz$$

Since $\cos 2\varphi=\cos^2\varphi-\sin^2\varphi=2\cos^2\varphi-1$ then

$\cos^2\varphi=\frac{1}{2}(1+\cos 2\varphi)$, giving

$$4^2\int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}-\cos 2\varphi+2\left(\tfrac{1}{2}(1+\cos 2\varphi)\right)^2\,d\varphi\,dz$$

$$= 4^2\int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}\left(-\cos 2\varphi+2\tfrac{1}{4}(1+2\cos 2\varphi+\cos^2 2\varphi)\right)d\varphi\,dz$$

$$= 4^2\int_{z=0}^{z=5}\int_{\varphi=0}^{\varphi=2\pi}\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{1}{2}(1+\cos 4\varphi)\,d\varphi\,dz$$

$$= 4^2\int_{z=0}^{z=5}\left.\tfrac{1}{2}\varphi+\tfrac{1}{4}\left(\varphi+\frac{\sin 4\varphi}{4}\right)\right)\Big]_{\varphi=0}^{\varphi=2\pi}dz$$

$$= 4^2\int_{z=0}^{z=5}\left(\tfrac{3}{4}2\pi+0-(0+0)\right)dz = 4\cdot 3\cdot 2\pi z\Big]_{z=0}^{z=5}$$

$$= 4\cdot 3\cdot 2\pi(5-0) = 5\cdot 4\cdot 3\cdot 2\pi. = 120\pi$$

So

$$\iint_S(x^3\hat{i}+y^3\hat{j})\cdot d\vec{S}=\iint_{S_1}(x^3\hat{i}+y^3\hat{j})\cdot d\vec{S}+\iint_{S_2}(x^3\hat{i}+y^3\hat{j})\cdot d\vec{S}+\iint_{S_3}(x^3\hat{i}+y^3\hat{j})\cdot d\vec{S}$$